Limitations of tensor-network approaches for optimization and sampling: A comparison to quantum and classical Ising machines

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Brief Introduction to Tensor Networks

Tensor

Definition

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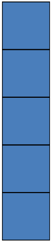
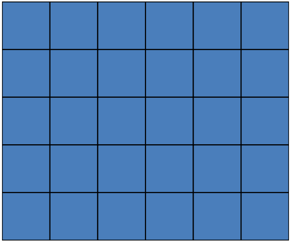
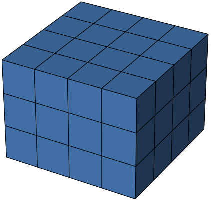


1st-order tensor
(vector of dimension [5])



2nd-order tensor
(matrix of dimensions [5,6])



3rd-order tensor
(tensor of dimension [4,4,3])

Tensor Contraction

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An **index contraction** is the sum over all the possible values of the repeated indices of a set of tensors.

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Example - matrix multiplication

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More complicated example

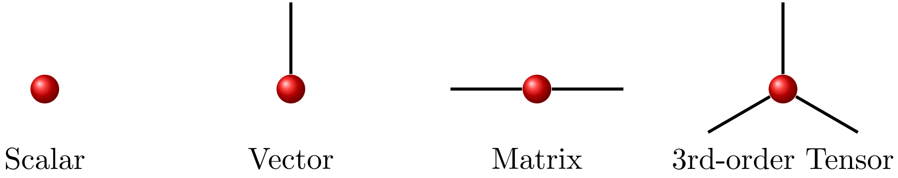
$$E = \sum_{i,j,k,l,m,n=1}^{D} A_{ijk}\, B_{ilm}\, C_{jln}\, D_{kmn}$$

Terminology

$$D_{ijk} = \sum_{l=1}^{D_1} \sum_{m=1}^{D_2} \sum_{n=1}^{D_3} A_{ljm} B_{iln} C_{nmk}.$$

- Indices that are not contracted are called **open indices**
- Contracted indices are called **bond** or **ancillary indices**
- Number of possible values D_i is referred to as **bond dimension**

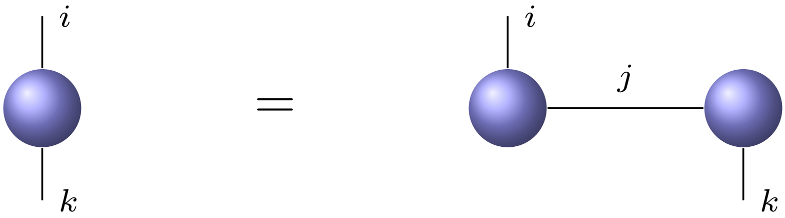
Graphical Notation



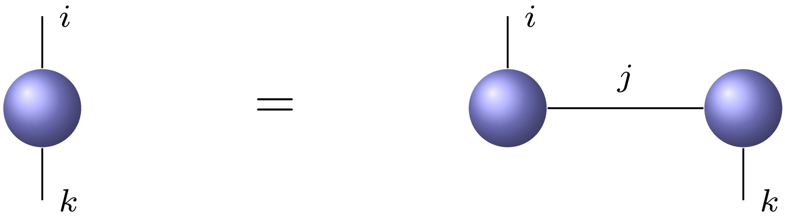
Shape

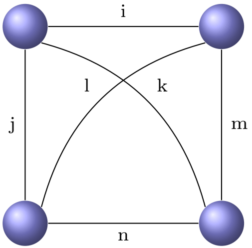
In general, the shape of tensors and the direction of lines carry no special significance.

Graphical Notation - Examples



Graphical Notation - Examples





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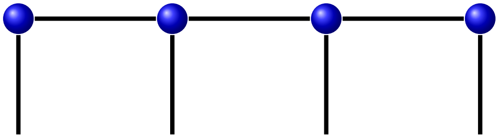
- The total number of operations depends heavily on the order in which indices in the TN are contracted.
- TN can be used to break complex systems (wave functions, partition functions, etc.) into smaller, more manageable pieces

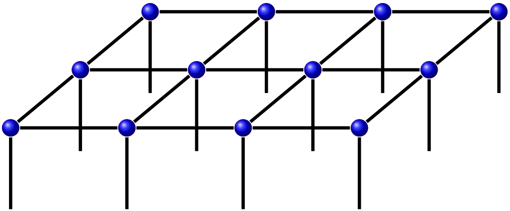
Matrix Product States (MPS)



Notes

- Basis for many powerful methods to simulate $1d$ quantum many-body systems (DMRG, TEBD, etc.).
- MPS can represent any quantum state of the many-body Hilbert space just by increasing sufficiently the value of the bond dimension (D).

Projected Entangled Pair States (PEPS)



Notes

- Natural generalization of MPS to higher spatial dimensions.
- In general, contracting PEPS exactly is #P-hard

Futher Reading

- Orús, R. (2014). A practical introduction to tensor networks: Matrix product states and projected entangled pair states. *Annals of physics*, 349, 117-158.
- Ran, S. J., Tirrito, E., Peng, C., Chen, X., Tagliacozzo, L., Su, G., & Lewenstein, M. (2020). *Tensor network contractions: methods and applications to quantum many-body systems*. Springer Nature.
- *https://tensornetwork.org/*
- *https://www.tensors.net/*

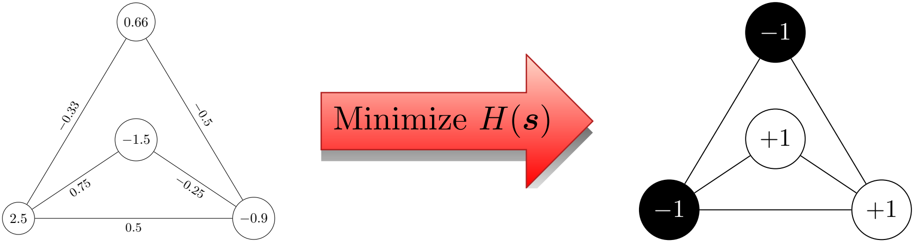
The Problem

The Optimization Problem

Finding The Ground State of The (classical) Ising Model

$$\min_{\underline{s}_N \in \{-1,1\}^N} H(\underline{s}_N) = \sum_{(i,j)\in E(G)} J_{ij} s_i s_j + \sum_{i=1}^{N} h_i s_i \quad (1)$$

where $\underline{s}_n = (s_1, \ldots, s_n)$, $s_i = \pm 1$, $J_{ij}, h_i \in \mathbb{R}$ and $G = (E, V)$ is a simple graph.



Why it is Interesting?

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1. Finding the ground state (minimal energy) of the Ising model with an arbitrary connectivity graph is a **NP-hard** problem.

Hardness of the Ising model

The only way to be guaranteed the correct answer is to check each possible spin configuration (*i.e.*, brute force), that is 2^N possibilities!

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2. There are many classical and quantum heuristic algorithms for solving this problem, including purpose-built machines, such as **Quantum Annealers (QA)**.

Quantum Annealers

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Quantum Annealing

$$\mathcal{H} = \underbrace{A(s)\left(\sum_{i=1}^{N} \sigma_i^x\right)}_{\text{Initial Hamiltonian}} + \underbrace{B(s)\left(\sum_{i=1}^{N} h_i \sigma_i^z + \sum_{(i,j)\in E(G)} J_{ij}\sigma_i^z\sigma_j^z\right)}_{\text{Final Hamiltonian}}, \quad (2)$$

where $s = \frac{t}{\tau}$ is normalized time, τ is total annealing time, $A(0) \gg B(0)$ and $A(1) \approx 0$, $A(1) \ll B(1)$, and σ^x, σ^z are Pauli matrices

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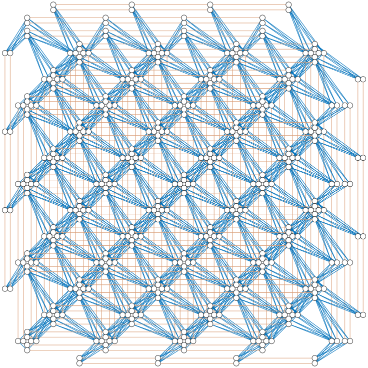
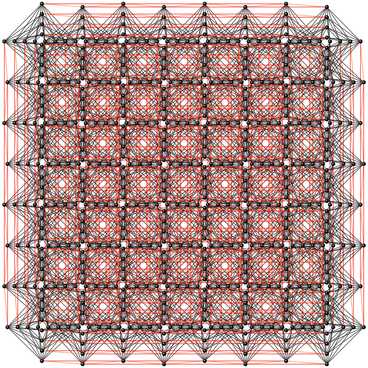
Important

Nonzero values of h_i and J_{ij} are limited to those available in the working graph!

Quantum Annealers' Quantum Processing Units (QPUs)



Pegasus topology
aprox. 5.5k qubits and 40k couplers



Zephyr topology
aprox. 7.4k qubits and 70k couplers

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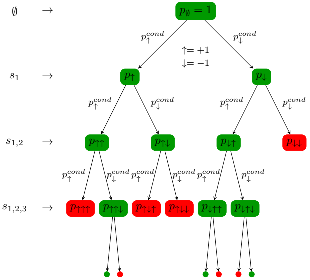
- Branch and Bound search in the probability space



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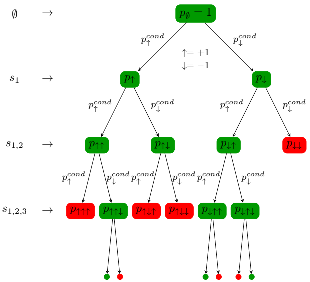

Problem Calculating marginal probabilities requires sampling from the Boltzmann distribution

$$p(\underline{s}_N) = \frac{e^{-\beta H(\underline{s}_N)}}{Z}$$

where Z is a partition function

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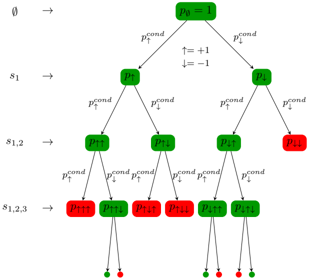

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Solution Use tensor networks to calculate Z

SpinGlassPEPS

A heuristic tensor-network-based algorithm to reveal the low-energy spectra of Ising spin-glass systems with interaction graphs relevant to present-day quantum annealers

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Three main parts:

1. Transform the Ising problem into one more suitable for QPU graphs and TN

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Three main parts:

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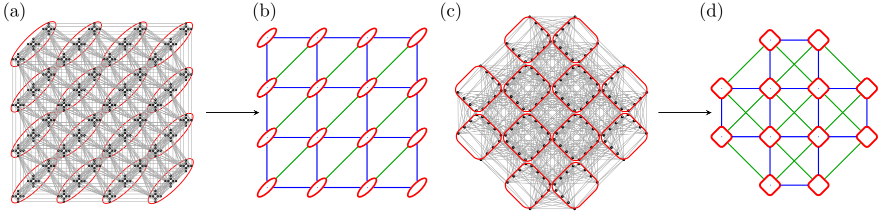
2. Perform the main sampling and search steps
3. Perform efficient tensor calculations

Tensor-Network Approach

The Generalized Potts Hamiltonian

$$H(\underline{x}_{\bar{N}}) = \sum_{(m,n)\in E(\mathcal{F})} E_{x_m x_n} + \sum_{n=1}^{\bar{N}} E_{x_n}$$

where $\mathcal{F}$ is a square lattice with $\bar{N}$ nodes, x_n are configuration variables, E_{x_n} intranode energy matrix, $E_{x_m x_n}$ iternode coupling energy matrix.



Size of Potts Variables

- x_n takes up to 2^{24} values for Pegasus graph
- x_n takes up to 2^{16} values for Zephyr graph

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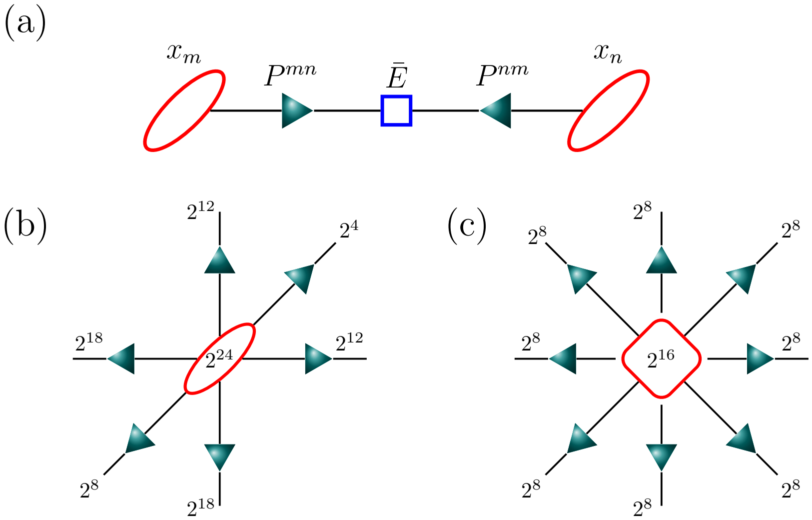
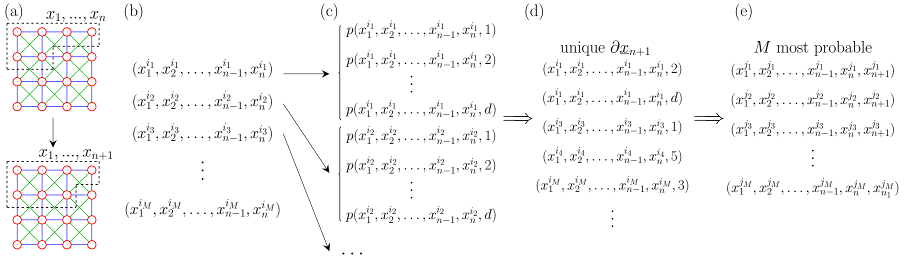
$$\bar{x}_m = P^{mn}(x_m),$$

where P^{mn} projecting d configurations in the mth node onto d^{mn} unique subconfigurations

Projectors Ilustrated



Branch and Bound



$$p(\underline{x}_{\bar{N}}) = \frac{e^{-\beta H(\underline{x}_{\bar{N}})}}{Z} \quad p(\underline{x}_{n+1}) = p(\underline{x}_{n+1}|\underline{x}_n) \times p(\underline{x}_n)$$

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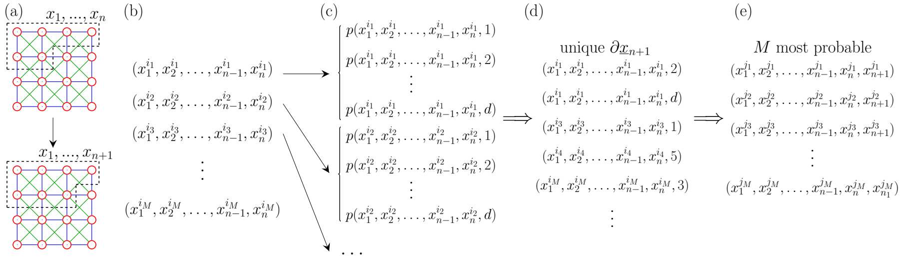

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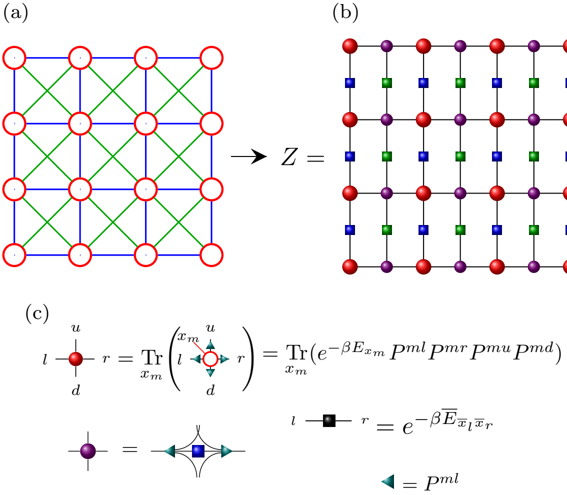
$$\partial \underline{x}_n = \{P^{kl}(\underline{x}_k) : \langle k, l \rangle \in \mathcal{F}, k \leqslant n < l\} \quad p(\underline{x}_{n+1}|\underline{x}_n) = p(\underline{x}_{n+1}|\partial \underline{x}_n)$$

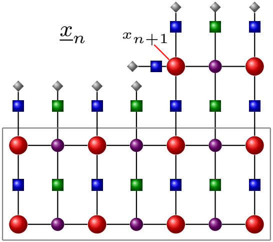
Tensor Network Construction



Calculation of conditional probabilities

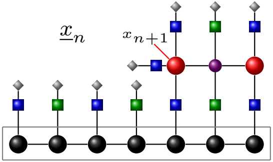
(a)

$$p(x_{n+1}|\underline{x}_n) =$$



(b)

$$\approx$$



Some Additional Optimizations

- Rotations

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- Sparse tensor structure
- Loopy belief propagation

Results

Benchmarking Setup I

Problem instances:

- No external magnetic fields ($h_i = 0$) and random coupling strenghts from $[-1, 1]$.
- Native instance for given QPU (class I)

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Metrics:

- Quality of solutions

$$d_E = \frac{(E - E_{\text{best}})}{2|E_{\text{best}}|}$$

- Diversity of solutions
  - approximation ratio $d_E \leqslant a_r$
  - relative distance threshold $d(\underline{s}_N, \underline{s'}_N) \geqslant RN$

Benchmarking Setup II

Solvers to compare

- **D-Wave** - commercially available quantum annealers

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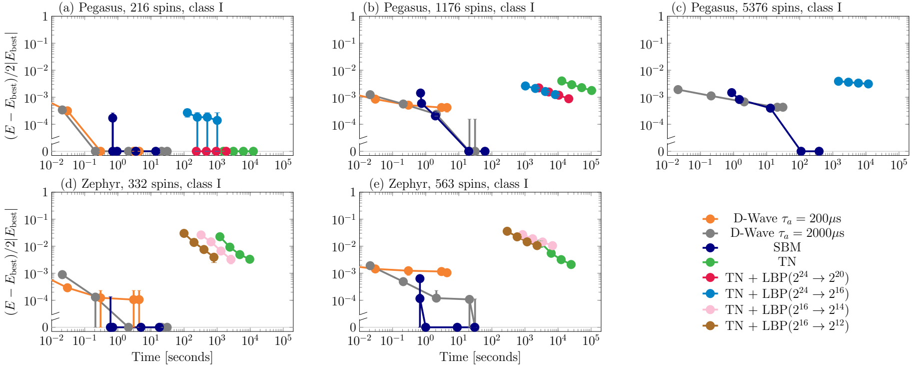
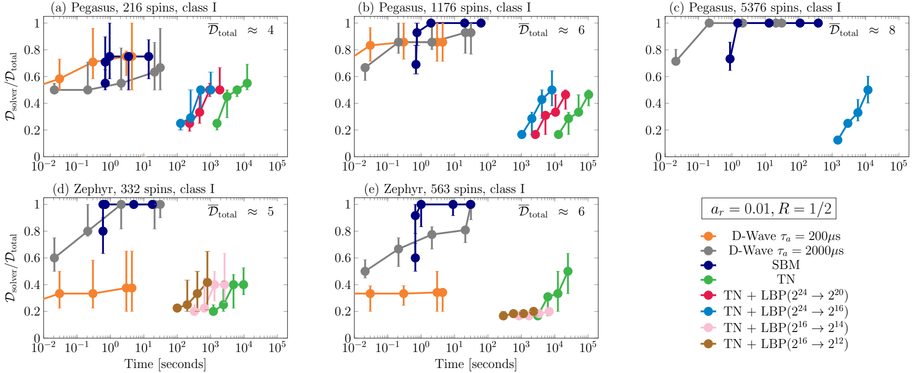
- **Simulated Bifurcation Machine (SBM)** - state-of-the-art physics-inspired solver

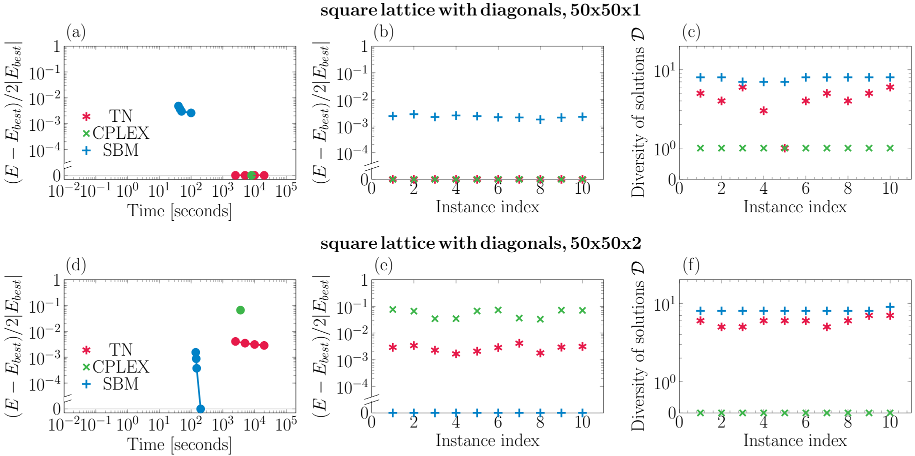
Class I - Quality of Solutions



Class I - Diversity of Solutions



Square lattice with diagonals



Conclusions

- Performance Trade-off

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- Sampling Limitations

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- Performance Trade-off
- Sampling Limitations
- Scalability and Stability Challenges

References

[1] Dziubyna, A. M., *et al.* "Limitations of tensor-network approaches for optimization and sampling: A comparison to quantum and classical Ising machines." *Physical Review Applied* 23.5 (2025).

[2] Śmierzchalski T., *et al.*, "SpinGlassPEPS.jl: Tensor-network package for Ising-like optimization on quasi-two-dimensional graphs" *arXiv preprint arXiv:2502.02317* (2025).

SpinGlassPEPS.jl Demo

Thank you for your attention!

Demo for SpinGlassPEPS.jl can be found here:
https://github.com/tomsmierz/demo