Primer on photonic quantum computing, Gaussian Boson Sampling and its applications

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Preface

This primer is prepared for the workshop "*Graph theoretical applications of photonic quantum computing*" held at IT4Innovations, Ostrava on September 2-3, 2025. They were compiled on a short timeline, and as such may contain errors or omissions. We kindly ask readers to notify us of any mistakes they encounter.

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Contents

1 Basics of photonic quantum computing **6**
- 1.1 Fock space and canonical commutation relations . . . 6
- 1.2 Elementary photonic quantum gates . . . 8
  - 1.2.1 Common photonic quantum gates . . . 9
- 1.3 Interferometers and their decomposition . . . 11
  - 1.3.1 Passive linear gates and the permanent . . . 11
  - 1.3.2 Clements decomposition . . . 12
- 1.4 Particle number measurement . . . 13
- 1.5 Boson Sampling . . . 13
- 1.6 Photon losses . . . 14
- 1.7 Exercises . . . 14

2 Gaussian states **17**
- 2.1 Elementary photonic quantum states . . . 17
- 2.2 Linear gates and the symplectic group . . . 18
- 2.3 Gaussian states . . . 20
- 2.4 Normal mode decomposition of Gaussian states . . . 21
- 2.5 Statistical moments of Gaussian states . . . 22
- 2.6 Bloch-Messiah decomposition . . . 23
- 2.7 Exercises . . . 24

3 Phase space formulation of quantum optics **27**
- 3.1 Weyl operators and coherent states . . . 27
- 3.2 Fourier-Weyl relation . . . 29
- 3.3 Characteristic functions and Wigner function . . . 30
  - 3.3.1 Wigner function . . . 31
  - 3.3.2 s-ordered characteristic functions . . . 32
- 3.4 Glauber-Sudarshan P-representation, Husimi Q-function . . . 33
- 3.5 Exercises . . . 34

4 Gaussian Boson Sampling **36**
- 4.1 Particle number detection probabilities of Gaussian states . . . 36
- 4.2 Properties of the hafnian and connection to graph theory . . . 37
  - 4.2.1 Recap: Graph theory . . . 37
  - 4.2.2 Perfect matchings and the hafnian . . . 38
- 4.3 Finding dense subgraphs using Gaussian Boson Sampling . . . 38
- 4.4 Exercises . . . 39

A Recap: Basics of quantum mechanics **42**
- A.1 Hilbert spaces . . . 42
- A.2 Quantum states . . . 43
- A.3 Evolution . . . 44
- A.4 Measurement . . . 44
- A.5 Composite systems . . . 45
- A.6 Quantum mechanics of the harmonic oscillator . . . 46

B First quantization **49**

Introduction

Quantum computers promise to be able to execute certain tasks that lie beyond the capabilities of classical computers [1, 2] with plenty of potential applications, ranging from many-body physics [3, 4] through quantum chemistry [5, 6, 7] problems to machine learning [8, 9]. This promise has recently spurred significant advancements in developing quantum computer prototypes, marking our entry into the second quantum revolution. Progress in quantum computing is ultimately driven by the many end users in industries. No wonder, that significant progress has been made by the private sector, both by large, established companies (e.g., Google [10], IBM [11], Microsoft [12], AWS [13]) and by smaller startups (e.g., Rigetti [14], Xanadu [15], PsiQuantum [16], Pasqal [17], Quix [18], Quantinuum [19], QuEra [20]).

However, developing a quantum computer comes with its unique set of challenges. The biggest obstacle is the suppression of decoherence, i.e., preserving the quantum coherence of the system when it interacts with its surroundings [21]. The ultimate goal is to construct quantum computers that are fault-tolerant. However, the present technology has not yet provided a scalable fault-tolerant quantum computer, and current devices are still noisy and not yet fully scalable [22]. Undeterred, one can regard these noisy intermediate-scale quantum (NISQ) computers as a significant step toward fault-tolerant quantum computing and as computers with potentially useful applications. Circuit sizes allowed by current hardware are already sufficient for non-trivial computations, as evidenced by quantum advantage experiments that cannot be simulated using conventional computers [23, 20].

Among the many proposals for quantum computation, the relevance of photonic quantum computers increased due to recent demonstrations of possible photonic quantum advantage schemes [24, 25, 15] and the development of feasible fault-tolerant quantum computation methods [16, 26, 27]. Photonic quantum computers are quantum computers that use photons as carriers of quantum information. Among the various quantum computing paradigms, photonic quantum computing stands out due to its unique advantages [28]: (i) Quantum phenomena (e.g., superposition, entanglement) can be observed in photons even at room temperature, while matter-based qubits require low temperatures. (ii) Relatively mature technologies can be used, e.g., beamsplitter and phaseshifter gates are already a crucial part of many optical experiments. (iii) Photons can be entangled over longer distances, enabling communication in a network of photonic quantum computers.

The study of graph theory has long provided a unifying mathematical framework for diverse fields such as computer science, physics, and chemistry. Photonic quantum computing, in particular, exhibits a deep connection to graph theory: many computational tasks can be rephrased in a graph-theoretic language, while photonic hardware natively realizes linear-optical networks that correspond to graph adjacency matrices. A key example of this interplay is *Boson Sampling*, introduced by Aaronson and Arkhipov [29], which can be understood as sampling from a probability distribution governed by the permanent of a submatrix of a unitary describing an optical interferometer. The permanent, in turn, enumerates perfect matchings of bipartite graphs. Similarly, *Gaussian Boson Sampling* (GBS) [30] extends this picture by connecting photon detection statistics to the hafnian and loop hafnian, functions that enumerate perfect matchings of graphs. These connections not only provide a natural connection between photonic quantum computing and graph theory, but also open a route to applying photonic devices for solving graph problems, such as finding dense subgraphs [31] or maximum cliques [32].

This primer is organized as follows. In Chapter 1 we review the theoretical basics of photonic quantum computing. Chapter 2 provides an introduction to (bosonic) Gaussian states, the quantum states playing a key role in Gaussian Boson Sampling. To illuminate the connection of Gaussian states with graph theory, we introduce the phase-space formalism of quantum optics in Chapter 3. Finally, in Chapter 4, we discuss Gaussian Boson Sampling and its applications in graph-theoretical problems.

Chapter 1

Basics of photonic quantum computing

The foundation of photonic quantum computing lies in the exploration of bosonic quantum systems. Thus, this chapter discusses the basics and introduces the main notions used throughout this material. For completeness, a brief recap on quantum systems in general can be found in Appendix A. Our discussion starts by using the occupation number representation directly, but the curious reader is advised to visit Appendix B to learn how occupation numbers are obtained from single-particle wave functions.

1.1 Fock space and canonical commutation relations

The basic object in photonic quantum systems we consider is the photonic *qumode* (or simply mode). Roughly speaking, in a single qumode, we can have a number of particles (photons) $n \geq 0$, which is represented by a vector written as $|n\rangle$, corresponding to the quantum state of qumode system of n photons. We call these vectors *Fock states*, n is referred to as the *occupation number*, and their generated vector space the (bosonic) *Fock space*. Fock states constitute an orthogonal basis for the Fock space, i.e.,

$$\langle n|m\rangle = \delta_{nm}, \tag{1.1}$$

where δ_{nm} is the Kroenecker delta, and it is 1 if $n = m$, 0 otherwise. The Fock state $|0\rangle$ is called the *vacuum state*. We define the *creation and annihilation operators* (collectively called *ladder operators*) by their action on the Fock states as

$$\begin{aligned} \hat{a}^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle\,, \\ \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \qquad (n \geq 1), \\ \hat{a}\,|0\rangle &= 0. \end{aligned} \tag{1.2}$$

The Fock state $|n\rangle$ can be constructed using the creation operator from the vacuum as

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{1.3}$$

Any single-mode pure state $|\psi\rangle$ can be decomposed using Fock states as

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle\,, \quad c_n \in \mathbb{C}, \tag{1.4}$$

where the normalization $\|\psi\| = 1$ implies that

$$\sum_{n=0}^{\infty} |c_n|^2 = 1. \tag{1.5}$$

The creation and the annihilation operators satisfy the following relation, which is easy to verify using Eq. (1.2):

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{1}, \tag{1.6}$$

where $\mathbb{1}$ is the identity operator and $[A, B] \coloneqq AB - BA$ is the *commutator*. Using the ladder operators, one can construct the *number operator* $\hat{n} \coloneqq \hat{a}^\dagger a$, which satisfies the following equation:

$$\hat{n} \left|n\right\rangle = \hat{a}^\dagger \hat{a} \left|n\right\rangle = n \left|n\right\rangle . \tag{1.7}$$

In the physical context, the vacuum state is interpreted as a quantum state with no photons. This may easily be understood by

$$\hat{n} \left|0\right\rangle = \sum_{i=1}^{d} \hat{a}_i^\dagger \hat{a}_i \left|0\right\rangle = 0, \tag{1.8}$$

since $\hat{a}_i \left|0\right\rangle = 0$ using the definition.

Generalizing the previous discussion, let us now consider a system with d modes. In this case, we have d photonic qumodes, and we write a generic Fock state as

$$\left|n_1, \ldots, n_d\right\rangle . \tag{1.9}$$

This corresponds to the quantum state where n_1 photon is in the first qumode, n_2 photon is in the second qumode, and so on. The creation and annihilation operators are modifed as

$$\begin{aligned} \hat{a}_j^\dagger \left|n_1, \ldots, n_j, \ldots, n_d\right\rangle &= \sqrt{n_j + 1} \left|n_1, \ldots, n_j + 1, \ldots, n_d\right\rangle , \\ \hat{a}_j \left|n_1, \ldots, n_j, \ldots, n_d\right\rangle &= \sqrt{n_j} \left|n_1, \ldots, n_j - 1, \ldots, n_d\right\rangle . \end{aligned} \tag{1.10}$$

For the creation and annihilation operators, the following relations hold:

$$\begin{aligned} [\hat{a}_i, \hat{a}_j^\dagger] &= \delta_{ij} \mathbb{1}, \\ [\hat{a}_i, \hat{a}_j] &= [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \end{aligned} \tag{1.11}$$

for all $i, j \in [d]$. These relations are called the *canonical commutation relations* (CCR), and the (associative) algebra generated by the ladder operators is called the *CCR algebra*. Elements of this algebra for which $t^\dagger = t$ are called self-adjoint.

Similarly as before, we call the generated vector state of (multimode) Fock states the Fock space. The Fock basis states can also be generated by applying creation operators to the vacuum state, i.e.,

$$\left|l_1, \ldots, l_d\right\rangle = \prod_{i=1}^{d} \frac{(\hat{a}_i^\dagger)^{l_i}}{\sqrt{l_i!}} \left|0\right\rangle . \tag{1.12}$$

These states are useful because they form a countable basis in the Fock space. Generally, any vector $\left|\psi\right\rangle$ corresponding to a (pure) state in the Fock space can be decomposed as

$$\left|\psi\right\rangle = \sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} c_{n_1,\ldots,n_d} \left|n_1, \ldots, n_d\right\rangle , \tag{1.13}$$

where $c_{n_1,\ldots,n_d} \in \mathbb{C}$ and

$$\sum_{n_1=0}^{\infty} \cdots \sum_{n_d=0}^{\infty} |c_{n_1,\ldots,n_d}|^2 = 1. \tag{1.14}$$

Instead of the ladder operators, one may equivalently use so-called *quadrature operators*, which are observables corresponding to the phase-space position and momentum.

Definition 1.1.1 (Quadrature operators)**.** The position operator $\hat{x}_i$ and momentum operator $\hat{p}_i$ are defined as

$$\begin{aligned} \hat{x}_i &= \frac{\hat{a}_i + \hat{a}_i^\dagger}{\sqrt{2}}, \\ \hat{p}_i &= \frac{\hat{a}_i - \hat{a}_i^\dagger}{i\sqrt{2}}, \end{aligned} \tag{1.15}$$

where $i \in [d]$. These operators are collectively called *quadrature operators*.

Note, that the quadrature operators are self-adjoint, and the CCR can also be expressed using the quadrature operators as

$$[\hat{x}_i, \hat{p}_j] = \delta_{ij}\mathbb{1}, \quad [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0, \tag{1.16}$$

for all $i, j \in [d]$. The transformation between ladder and quadrature operators can be written in matrix notation as

$$\boldsymbol{\xi} = W\boldsymbol{\chi}, \tag{1.17}$$

where the operators are collected in a vector as

$$\begin{aligned} \boldsymbol{\xi} &= [\hat{a}_1, \dots, \hat{a}_d, \hat{a}_1^\dagger, \dots, \hat{a}_d^\dagger]^T, \\ \boldsymbol{\chi} &= [\hat{x}_1, \dots, \hat{x}_d, \hat{p}_1, \dots, \hat{p}_d]^T, \\ W &= \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbb{1} & i\mathbb{1} \\ \mathbb{1} & -i\mathbb{1} \end{bmatrix}. \end{aligned} \tag{1.18}$$

1.2 Elementary photonic quantum gates

In photonic quantum computing, a quantum circuit can be described by a unitary operator over the Fock space. The elementary building blocks of photonic quantum circuits are called elementary photonic quantum gates, which can be used to construct a wide range of unitary operators on the bosonic Fock space. Elementary photonic quantum gates can be characterized by their photonic Hamiltonians, which are self-adjoint operators; the photonic gates can be written by exponentiating these, see Eq. (A.8).

In this section we present a useful categorization for photonic quantum gates. One important aspect is the degree of the Hamiltonian as a polynomial in the ladder operators, which characterizes their dynamics. In particular, we will see later, that photonic gates with degree 2 Hamiltonians have a particularly nice characterization. Another useful distinguishing feature of photonic quantum gates concerns the particle number preservation, since particle number preserving photonic quantum gates have the property that their unitary operators on the bosonic Fock space can be decomposed into direct sums corresponding to the particle number.

In the context of linear optics, photonic gates correspond to Hamiltonian operators that are at most quadratic in the ladder operators. For this purpose we introduce the following terminology:

Definition 1.2.1 (Quadratic Hamiltonian)**.** A d-mode quadratic Hamiltonian $\hat{H}$ is a self-adjoint operator of the form

$$\hat{H} = \boldsymbol{\xi}^\dagger H \boldsymbol{\xi} = \sum_{i,j=1}^{2d} H_{ij}\xi_i^\dagger \xi_j, \tag{1.19}$$

where $\boldsymbol{\xi}$ is defined by Eq. (1.18), $\boldsymbol{\xi}^\dagger$ is understood as an elementwise adjoint,

$$H = \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix}, \tag{1.20}$$

and $A, B \in \mathbb{C}^{d\times d}$ such that $A^\dagger = A$ and $B^T = B$. We will call the matrix H the *Hamiltonian matrix*.

In quadrature basis, one can also write quadratic Hamiltonians as

$$\hat{H} = \boldsymbol{\chi}^T H^{(\mathrm{xp})} \boldsymbol{\chi}, \tag{1.21}$$

where $H^{(\mathrm{xp})} \in \mathbb{R}^{2d\times 2d}$ is the Hamiltonian matrix in the quadrature basis, and has the form

$$H^{(\mathrm{xp})} = W^\dagger H W = \begin{bmatrix} \mathrm{Re}(A+B) & -\mathrm{Im}(A-B) \\ \mathrm{Im}(A+B) & \mathrm{Re}(A-B) \end{bmatrix}. \tag{1.22}$$

Definition 1.2.2 (Linear optical Hamiltonian)**.** A linear optical Hamiltonian can be written as

$$\hat{H} = \boldsymbol{\xi}^\dagger H \boldsymbol{\xi} + \breve{\boldsymbol{\alpha}} \cdot \boldsymbol{\xi}, \tag{1.23}$$

where $\boldsymbol{\xi}^\dagger H \boldsymbol{\xi}$ is a quadratic Hamiltonian as described in Definition 1.2.1, $\breve{\boldsymbol{\alpha}} := [\boldsymbol{\alpha}, \overline{\boldsymbol{\alpha}}]$ and $\boldsymbol{\alpha} \in \mathbb{C}^d$.

A central notion in linear optics is the concept of *linear optical gates*, which are photonic quantum gates with linear optical Hamiltonians. These gates are called *linear* because they act (affine) linearly on the ladder operators. Photonic quantum gates that are not linear are called *non-linear*, but we will not use this in this workshop. Another important categorization of the photonic quantum gates can be made based on how they change the number of particles in our system. Gates that do not alter the number of particles are called *passive gates*, and gates that modify the number of particles are called *active gates* [33]. More specifically, let us consider the *global particle number operator*

$$\hat{n} \coloneqq \sum_{j=1}^{d} \hat{n}_j = \sum_{j=1}^{d} \hat{a}_j^\dagger \hat{a}_j. \tag{1.24}$$

Passive gates are defined by preserving the global particle number operator:

Definition 1.2.3 (Passive gate)**.** A *passive gates* is a photonic quantum gate with unitary operator U commuting with the global particle number operator, i.e.,

$$[U, \hat{n}] = 0. \tag{1.25}$$

1.2.1 Common photonic quantum gates

In this section, the most important examples of photonic gates for this thesis are provided. Here, not only the unitary operator on the Fock space will be presented, but also the evolution of the creation and annihilation operators by these gates. This is important, since—as we will see—in the theory of linear optics, the unitary evolution of creation and annihilation operators can help characterize the evolution of photonic quantum states. To calculate this, the Hadamard lemma[1] needs to be used [34]. Considering a photonic Hamiltonian $\hat{H}$ and an operator X, the Hadamard lemma may be written as

$$e^{i\hat{H}} X e^{-i\hat{H}} = e^{\mathrm{ad}_{i\hat{H}}} X = X + [i\hat{H}, X] + \frac{1}{2!}[i\hat{H}, [i\hat{H}, X]] + \frac{1}{3!}[i\hat{H}, [i\hat{H}, [i\hat{H}, X]]] + \cdots . \tag{1.26}$$

The *phaseshift (or rotation) gate* models the actual phase shifter optical element, which rotates the phase of a traveling electromagnetic wave. As a quantum gate, the phaseshift gate acts on a state by rotating it in the canonical phase space.

Definition 1.2.4 (Phaseshift gate)**.** The *phaseshift gate* is defined on a single mode, and its operator is

$$R_j(\phi) = \exp\left(i\phi\hat{n}_j\right), \tag{1.27}$$

where $\phi \in [0, 2\pi)$ and $\hat{n}_j \coloneqq \hat{a}_j^\dagger \hat{a}_j$.

Importantly, the phaseshift gate is the only single-mode passive linear optical gate, therefore it is considered as an elementary building block of photonic quantum circuits. The phaseshift gate transforms the ladder operators in the Heisenberg picture as follows:

$$\begin{aligned} R_j(\phi)^\dagger \hat{a}_j R_j(\phi) &= \hat{a}_j e^{i\phi}, \\ R_j(\phi)^\dagger \hat{a}_j^\dagger R_j(\phi) &= \hat{a}_j^\dagger e^{-i\phi}, \end{aligned} \tag{1.28}$$

which can be derived using Eq. (1.26). One can write the previous expression more succinctly as

$$R_j^\dagger(\phi) \begin{bmatrix} \hat{a}_j \\ \hat{a}_j^\dagger \end{bmatrix} R_j(\phi) = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{bmatrix} \begin{bmatrix} \hat{a}_j \\ \hat{a}_j^\dagger \end{bmatrix}. \tag{1.29}$$

The phaseshift gates together with the *beamsplitter gates* generate all the passive linear optical transformations, which will become clear later in this Section 1.3.2. A beamsplitter is an optical device that splits light into two parts and is an essential part of many optical experiments. Importantly, beamsplitters play an essential role also in quantum teleportation, quantum computing, and the fundamental studies of photons.

Definition 1.2.5 (Beamsplitter gate)**.** Consider a bosonic system with d modes. In terms of the ladder operators, the unitary operator corresponding to the *beamsplitter gate* between modes labeled by $i, j \in [d] \coloneqq \{1, \ldots, d\}$ is

$$B_{jk}(\theta, \phi) = \exp\left(\theta(e^{i\phi}\hat{a}_j\hat{a}_k^\dagger - e^{-i\phi}\hat{a}_j^\dagger\hat{a}_k)\right), \tag{1.30}$$

where $\theta, \phi \in [0, 2\pi)$ are the angle parameters of the beamsplitter. When $\theta = \pi/4$ and $\phi = 0$, it is called a 50/50 beamsplitter.

[1] Also called the Campbell identity.

The ladder operators are evolved under this gate as

$$B_{jk}^{\dagger}(\theta,\phi)\begin{bmatrix}\hat{a}_j\\ \hat{a}_k\\ \hat{a}_j^{\dagger}\\ \hat{a}_k^{\dagger}\end{bmatrix}B_{jk}(\theta,\phi)=\begin{bmatrix}t & -\overline{r} & & \\ r & t & & \\ & & t & -r\\ & & \overline{r} & t\end{bmatrix}\begin{bmatrix}\hat{a}_j\\ \hat{a}_k\\ \hat{a}_j^{\dagger}\\ \hat{a}_k^{\dagger}\end{bmatrix}, \tag{1.31}$$

where $t=\cos(\theta)$ and $r=e^{i\phi}\sin(\theta)$.

Remark. Let us denote the two input fields with $\hat{a}_0,\hat{a}_1$, and the two output fields with $\hat{b}_0,\hat{b}_1$. The input and output states are linearly related by

$$\begin{bmatrix}\hat{b}_0\\ \hat{b}_1\end{bmatrix}=\begin{bmatrix}t' & r\\ r' & t\end{bmatrix}\begin{bmatrix}\hat{a}_0\\ \hat{a}_1\end{bmatrix}. \tag{1.32}$$

The canonical commutation relations (1.11) yield

$$|t'|=|t|,\quad |r'|=|r|,\quad |t|^2+|r|^2=1, \tag{1.33}$$

where t and r are called the transmittance and reflectance, respectively.

The *squeezing gate* is the prototypical example of an active linear optical gate.

Definition 1.2.6 (Squeezing gate)**.** The unitary operator corresponding to the *squeezing gate* is

$$S_j(z)=\exp\left(\frac{1}{2}\left(\overline{z}\hat{a}_j^2-z\hat{a}_j^{\dagger 2}\right)\right), \tag{1.34}$$

where $z\in\mathbb{C}$.

The name of the squeezing gate stems from its operational interpretation in the phase space. Considering a vacuum state as the initial state, the squeezing gate produces a state that is "squeezed" in the phase space along a certain direction[2]. This quantum state is called the squeezed vacuum state, or simply just squeezed state. Moreover, the squeezing gate transforms the ladder operators as

$$S_j^{\dagger}(z)\begin{bmatrix}\hat{a}_j\\ \hat{a}_j^{\dagger}\end{bmatrix}S_j(z)=\begin{bmatrix}\cosh r & -e^{i\phi}\sinh r\\ -e^{-i\phi}\sinh r & \cosh r\end{bmatrix}\begin{bmatrix}\hat{a}_j\\ \hat{a}_j^{\dagger}\end{bmatrix}, \tag{1.35}$$

where $z=re^{i\phi}$. In terms of the quadrature operators for $\phi\equiv 0$, this relation becomes

$$S_j^{\dagger}(z)\begin{bmatrix}\hat{x}_j\\ \hat{p}_j\end{bmatrix}S_j(z)=\begin{bmatrix}e^{-r} & 0\\ 0 & e^{r}\end{bmatrix}\begin{bmatrix}\hat{x}_j\\ \hat{p}_j\end{bmatrix}. \tag{1.36}$$

By analyzing this relation further, one can deduct that by applying a squeezing gate on a photonic quantum state, the variance of the expectation value of $\hat{x}_j$ gets smaller, justifying the term "squeezing".

Another notable active linear gate is the *displacement gate*, which has similar operational meaning in the phase space as the squeezing gate. As the name indicates, the displacement gate corresponds to "displacing" the quantum state in the phase space.

Definition 1.2.7 (Displacement gate)**.** The unitary operator corresponding to the *displacement gate* is

$$D_j(\alpha)=\exp\left(\alpha\hat{a}_j^{\dagger}-\overline{\alpha}\hat{a}_j\right), \tag{1.37}$$

where $\alpha\in\mathbb{C}$.

The displacement gate transforms the ladder operators as

$$D_j^{\dagger}(\alpha)\begin{bmatrix}\hat{a}_j\\ \hat{a}_j^{\dagger}\end{bmatrix}D_j(\alpha)=\begin{bmatrix}\hat{a}_j+\alpha\mathbb{1}\\ \hat{a}_j^{\dagger}+\overline{\alpha}\mathbb{1}\end{bmatrix}. \tag{1.38}$$

However, for more intuition, one can consider transforming the quadrature operators $\hat{x}_j$ and $\hat{p}_j$ from Definition 1.1.1, i.e.,

$$D_j^{\dagger}(\alpha)\begin{bmatrix}\hat{x}_j\\ \hat{p}_j\end{bmatrix}D_j(\alpha)=\begin{bmatrix}\hat{x}_j+\sqrt{2}\,\mathrm{Re}(\alpha)\mathbb{1}\\ \hat{p}_j+\sqrt{2}\,\mathrm{Im}(\alpha)\mathbb{1}\end{bmatrix}. \tag{1.39}$$

[2]More precisely, the Wigner function corresponding to the state is "squeezed".

This means, that the parameter $\alpha \in \mathbb{C}$ describes the shift of the quadrature operators in the two-dimensional phase space, i.e., the real part is responsible for a shift in position, and the imaginary part for a shift in momentum. More generally, considering a circuit consisting of d modes, a column of displacement gates parametrized by $\boldsymbol{\alpha} \in \mathbb{C}^d$ can describe a general shift of all quadrature operators in the phase space, providing an important tool for studying quantum optics.

1.3 Interferometers and their decomposition

The description of the phaseshifter and beamsplitter gates can be unified by discussing a generic optical *interferometer*. The interferometer represents an arbitrary passive linear operation, as it will become clear later in this chapter.

Definition 1.3.1 (Interferometer)**.** The unitary operator corresponding to an interferometer can be written as

$$I(U) = \exp\Big(i\hat{H}\Big), \tag{1.40}$$

where $\hat{H}$ is a quadratic Hamiltonian

$$\hat{H} = \boldsymbol{\xi}^\dagger H \boldsymbol{\xi}, \qquad H = \frac{1}{2}\begin{bmatrix} A & \\ & \overline{A} \end{bmatrix}, \tag{1.41}$$

where A is self-adjoint. The interferometer is often parametrized as $U = e^{iA}$, which will be justified in Section 1.3.1.

1.3.1 Passive linear gates and the permanent

Let us consider a d-mode bosonic system with one-particle Hilbert space $\mathcal{H}$. Moreover, consider a generic passive linear optical gate with a corresponding unitary operator in the form of

$$\hat{U} = \exp\Big(i\hat{H}\Big), \qquad \hat{H} = \boldsymbol{\xi}^\dagger H \boldsymbol{\xi}, \qquad H = \frac{1}{2}\begin{bmatrix} A & \\ & \overline{A} \end{bmatrix}, \tag{1.42}$$

as described in Definition 1.3.1. We will see later, that the transformation of the ladder operators can be written as

$$\begin{aligned} \hat{U}^\dagger \hat{a}_i \hat{U} &= \sum_{j=1}^{d} U_{ij} \hat{a}_j, \\ \hat{U}^\dagger \hat{a}_i^\dagger \hat{U} &= \sum_{j=1}^{d} \overline{U_{ij}} \hat{a}_j^\dagger, \end{aligned} \tag{1.43}$$

or more succinctly, as

$$\hat{U}^\dagger \boldsymbol{\xi} \hat{U} = \begin{bmatrix} U & \\ & \overline{U} \end{bmatrix} \boldsymbol{\xi}, \tag{1.44}$$

where $U = e^{iA}$. This means, that the creation/annihilation operators are mapped to a linear combination of creation/annihilation operators, respectively. Moreover, any passive linear optical gate $\hat{U}$ stabilizes the vacuum state, i.e.,

$$\hat{U} \left|0\right\rangle = \left|0\right\rangle. \tag{1.45}$$

One can easily calculate how such operator acts on single particles, since

$$\hat{U} \left|e_i\right\rangle = \hat{U}\hat{a}_i^\dagger \hat{U}^\dagger \underbrace{\hat{U} \left|0\right\rangle}_{=\left|0\right\rangle} = \sum_{j=1}^{d} U_{ji} \hat{a}_j^\dagger \left|0\right\rangle = U \left|e_i\right\rangle, \tag{1.46}$$

where $\left|e_i\right\rangle$ is the the one-particle basis vector

$$\left|0 \ldots \underbrace{1}_{i\text{-th position}} \ldots 0\right\rangle =: \left|e_i\right\rangle. \tag{1.47}$$

More generally, let us consider an n-particle Fock basis state $|\boldsymbol{n}\rangle \coloneqq |n_1, \ldots, n_d\rangle$. Passive linear optical gates act on each particle independently, which is illuminated by the following:

$$\hat{U}\,|\mathbf{n}\rangle = \hat{U}\prod_{i=1}^{d}\frac{(\hat{a}_i^\dagger)^{n_i}}{\sqrt{n_i!}}\,|0\rangle = \prod_{i=1}^{d}\frac{(\hat{U}\hat{a}_i^\dagger\hat{U}^\dagger)^{n_i}}{\sqrt{n_i!}}\,|0\rangle = \prod_{i=1}^{d}\frac{(\sum_{j=1}^{d}U_{ji}\hat{a}_j^\dagger)^{n_i}}{\sqrt{n_i!}}\,|0\rangle\,. \tag{1.48}$$

Here, we used that passive linear gates leave the vacuum state unchanged, i.e., $\hat{U}\,|0\rangle = |0\rangle$. Now, expand each factor using the multinomial theorem, and introduce nonnegative integers m_{ji} counting how many times mode j is chosen from input i:

$$\Big(\sum_{j=1}^{d}U_{ji}\hat{a}_j^\dagger\Big)^{n_i} = \sum_{m_{1i}+\cdots+m_{di}=n_i}\frac{n_i!}{\prod_j m_{ji}!}\prod_{j=1}^{d}\left(U_{ji}\hat{a}_j^\dagger\right)^{m_{ji}}. \tag{1.49}$$

Multiplying over all inputs i gives a sum over the whole nonnegative matrix $M = (m_{ji})_{j,i}$ with row constraints $\sum_{i=1}^{d} m_{ji} =: m_j$ and column constraints $\sum_{j=1}^{d} m_{ji} = n_i$.

Since the $\hat{a}_j^\dagger$'s commute, we can write

$$\prod_{i=1}^{d}\sum_{m_{1i}+\cdots+m_{di}=n_i}\frac{n_i!}{\prod_j m_{ji}!}\prod_{j=1}^{d}\left(U_{ji}\hat{a}_j^\dagger\right)^{m_{ji}} = \sum_{\substack{m_{ij}\geq 0\\ \sum_i m_{ji}=m_j\\ \sum_j m_{ji}=n_i}}\prod_{i,j=1}^{d}\frac{n_i!}{\prod_j m_{ji}!}U_{ji}^{m_{ji}}(\hat{a}_j^\dagger)^{m_j} \tag{1.50}$$

and hence, when $\{m_{ji}\}$ satisfy the row constraint for $|\boldsymbol{m}\rangle = |m_1, \cdots, m_d\rangle$, we can write

$$\langle\boldsymbol{m}|\,\hat{U}\,|\boldsymbol{n}\rangle = \sum_{\substack{m_{ij}\geq 0\\ \sum_i m_{ji}=m_j\\ \sum_j m_{ji}=n_i}}\prod_{i,j=1}^{d}\frac{\sqrt{n_i!m_j!}}{\prod_j m_{ji}!}U_{ji}^{m_{ji}}. \tag{1.51}$$

Now we need to use the following definition:

Definition 1.3.2. The *permanent* of a matrix $A \in \mathbb{C}^{n\times n}$ is defined as

$$\mathrm{per}(A) = \sum_{\sigma\in S_n}\prod_{i=1}^{n}A_{\sigma(i),i}, \qquad A\in\mathbb{C}^{n\times n}, \tag{1.52}$$

where S_n denotes the set of permutations of n elements.

We can easily see that

$$\mathrm{per}(\mathrm{r}_{\boldsymbol{m},\boldsymbol{n}}\,U) = \sum_{\substack{m_{ij}\geq 0\\ \sum_i m_{ji}=m_j\\ \sum_j m_{ji}=n_i}}\prod_{i,j=1}^{d}\frac{n_i!m_j!}{\prod_j m_{ji}!}U_{ji}^{m_{ji}}, \tag{1.53}$$

where $\mathrm{r}_{\boldsymbol{m},\boldsymbol{n}}$ is the *matrix reduction*, denoting the operation of repeating the rows and columns of the input matrix according to $\boldsymbol{m}$ and $\boldsymbol{n}$, respectively. Hence, the overlap is

$$\langle\boldsymbol{m}|\,\hat{U}\,|\boldsymbol{n}\rangle = \frac{\mathrm{per}(\mathrm{r}_{\boldsymbol{m},\boldsymbol{n}}\,U)}{\sqrt{\boldsymbol{n}!\boldsymbol{m}!}}, \tag{1.54}$$

where we denoted $\boldsymbol{n}! = \prod_{i=1}^{n} n_i!$.

1.3.2 Clements decomposition

It is a natural question to ask for a decomposition of any passive linear operation (i.e., any interferometer) into elementary building blocks such as beamsplitters and phaseshifters, since interferometers are not directly realizable on a photonic quantum computer in general. Moreover, several applications of photonic quantum computing require a tangible parametrization of photonic quantum circuits, where such decomposition is invaluable.

The Clements decomposition provides a way to decompose any passive linear transformation into beamsplitters and phaseshifters [35]. An illustration of the decomposition can be seen in Figure 1.1. The idea is based on iterative applications of Givens rotations onto the one-particle unitary matrix corresponding to the interferometer, where each Givens rotation corresponds to a beamsplitter. With this idea, one can eliminate all elements from the unitary matrix one-by-one, yielding a diagonal matrix corresponding to phaseshifters.

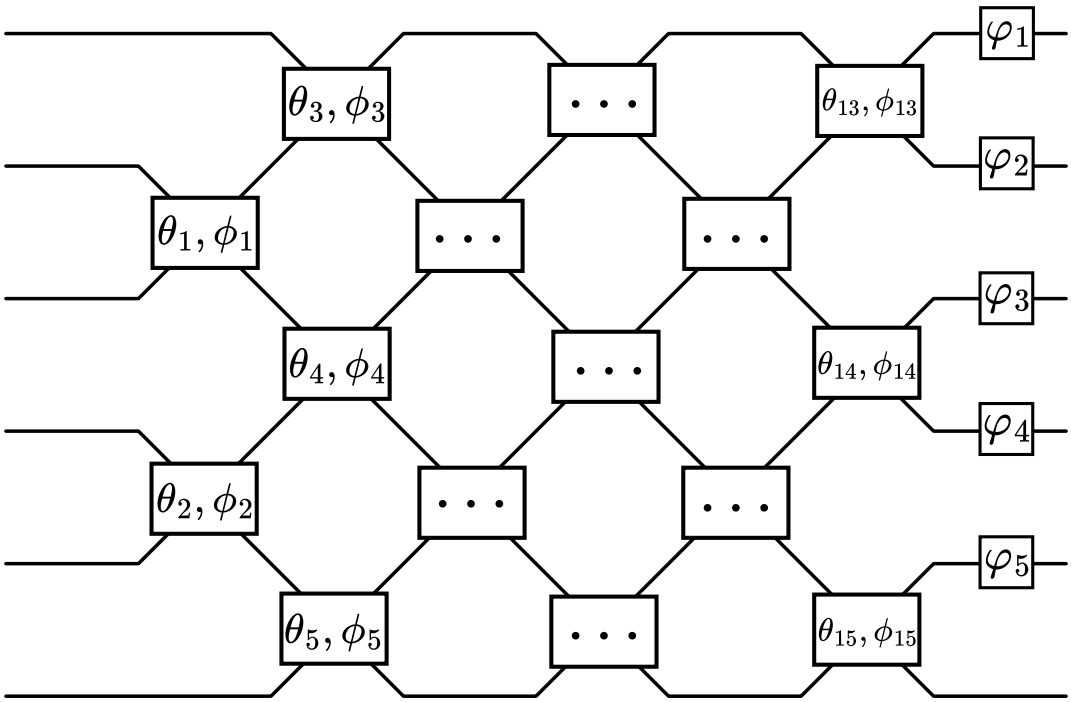

Figure 1.1: Illustration of the Clements decomposition for 6 modes. The two-mode gates denote beamsplitters, while the single-mode gates denote phaseshifters. Generally, a d-mode interferometer is decomposed into $\frac{d(d-1)}{2}$ beamsplitters and $d-1$ phaseshifters. Note that the last phaseshifter is omitted due to the global phase invariance.

1.4 Particle number measurement

Particle number measurement or photon detection is a measurement that is implemented via *photon number-resolving detectors* (PNRDs) experimentally. The probability of detecting particle numbers $\boldsymbol{n} = (n_1, n_2, \ldots, n_d)$ in a pure quantum state $|\psi\rangle$ is given by

$$p(\mathbf{n}) = |\langle \boldsymbol{n}|\psi\rangle|^2. \tag{1.55}$$

The obtained samples from the measurement are non-negative integer values corresponding to the detected photon number, hence, the probabilites can only be approximated after taking multiple samples. Notice, that Eq. (1.55) is consistent with Eq. (1.14), since

$$|\langle \boldsymbol{n}|\psi\rangle|^2 = |c_{\boldsymbol{n}}|^2 = |c_{n_1,\ldots,n_d}|^2. \tag{1.56}$$

1.5 Boson Sampling

One notable quantum advantage scheme that can be implemented on a photonic quantum computer is the well-known Boson Sampling scheme [29]. Boson Sampling is a quantum computing approach that involves generating samples via particle number measurement (see Section 1.4) from a photonic quantum state prepared by an interferometer starting from a Fock basis state. Although this setup is theoretically applicable to any bosonic system, the photonic implementation is the most natural. It is generally accepted that Boson Sampling executes a computational task more efficiently than classical computers because solving it requires sampling from the output particle number probability distribution with a randomly chosen interferometer, a challenge considered intractable for classical computation. Specifically, the difficulty of this problem has been demonstrated in the *collision-free* regime, with number of modes scaling quadratically in the number of input photons [29]. While Boson Sampling is primarily designed for quantum advantage demonstrations without any practical application in mind, this scheme has proposed applications for solving electronic structure problems [36] or determining molecular vibronic spectra [5].

In the standard Boson Sampling setup, the input state is typically a Fock basis state $|\boldsymbol{n}\rangle = |n_1, n_2, \ldots n_d\rangle$, where $n = \sum_{i=1}^{d} n_i$ denotes the total number of photons. Although $|\boldsymbol{n}\rangle$ can be arbitrary, the values of n_i are typically chosen to be either 1 or 0. The photons are then passed through a generic passive linear optical interferometer, characterized by a $d \times d$ unitary matrix U. Since a passive linear circuit preserves the total number of particles, the

resulting state also has n total particles. At the end of the circuit, photon detection is performed on each mode, and the probability of obtaining a specific measurement outcome $|\boldsymbol{m}\rangle = |m_1, m_2, \ldots, m_d\rangle$ is given by

$$p(\boldsymbol{m}) = \frac{|\mathrm{per}(\mathrm{r}_{\boldsymbol{m},\boldsymbol{n}}(U))|^2}{n_1! \ldots n_d! m_1! \ldots m_d!}. \tag{1.57}$$

Here, U is the unitary matrix corresponding to the passive linear circuit, and the permanent per of a matrix is defined via Eq. (1.52). Most importantly, the permanent is the source of the widely accepted classical intractability of the Boson Sampling problem: given a general matrix $A \in \mathbb{C}^{n\times n}$ even approximating $\mathrm{per}(A)$ is $\#P$-complete, and finding a polynomial-time algorithm for calculating this would imply P = NP [29].

1.6 Photon losses

Photon losses are one of the central challenges in photonic quantum computing, arising from absorption, scattering, or detector inefficiencies. Such losses reduce the computational power of photonic processors, making their mitigation a key requirement for scalable experiments. Since all current implementations are subject to non-negligible loss, it is essential to develop accurate analytic and numerical models of loss channels.

Photon losses are usually modeled as an interaction with an inaccessible environment mode (typically modeled by a vacuum state) via a beamsplitter: If a photon has survival probability η, the mode is coupled to an environmental vacuum state with transmission amplitude $|t| = \sqrt{\eta}$ and reflection amplitude $|r| = \sqrt{1-\eta}$. This is related to the beamsplitter angle as $\cos\theta = \sqrt{\eta}$ and for simplicity we choose $\phi = \pi/2$. The creation operator under this operation is transformed as

$$\hat{a}^\dagger \mapsto \sqrt{\eta}\hat{a}^\dagger + \sqrt{1-\eta}\hat{a}^\dagger_{\mathrm{env}}, \tag{1.58}$$

and since $\hat{a}^\dagger$ and $\hat{a}^\dagger_{\mathrm{env}}$ commute, we can write

$$\begin{aligned} |n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} &\mapsto \frac{\left(\sqrt{\eta}\hat{a}^\dagger + \sqrt{1-\eta}\hat{a}^\dagger_{\mathrm{env}}\right)^n}{\sqrt{n!}} |0, 0_{\mathrm{env}}\rangle \\ &= \frac{1}{\sqrt{n!}} \sum_{k=0}^{n} \binom{n}{k} (\sqrt{\eta})^k (\sqrt{1-\eta})^{n-k} (\hat{a}^\dagger)^k (\hat{a}^\dagger_{\mathrm{env}})^{n-k} |0, 0_{\mathrm{env}}\rangle \\ &= \sum_{k=0}^{n} \sqrt{\binom{n}{k}} (\sqrt{\eta})^k (\sqrt{1-\eta})^{n-k} |k, (n-k)_{\mathrm{env}}\rangle\,, \end{aligned} \tag{1.59}$$

where we denoted the environmental modes in the Fock states for clarity.

1.7 Exercises

Exercise 1.7.1. Install `piquasso` version 6.2.0 using

```
pip install piquasso
```

Verify that the installation is successful by opening a Python terminal and importing it!

Exercise 1.7.2. Using Piquasso, prepare a Fock state $|11\rangle$ using `SamplingSimulator`! Use `ParticleNumberMeasurement` to simulate photon detection, and obtain samples.

Solution. The Piquasso code is:

```
import piquasso as pq

with pq.Program() as program:
    pq.Q() | pq.StateVector(occupation_numbers=(1, 1)) | pq.ParticleNumberMeasurement()

simulator = pq.SamplingSimulator(d=2)

result = simulator.execute(program, shots=20)

print(result.samples)

```

Exercise 1.7.3. Apply a 50/50 beamsplitter on the prepared state from the previous exercise, and apply photon detection! Estimate the probabilities of the output patterns!

Solution. The Piquasso code is:

```
import numpy as np
import piquasso as pq

shots = 100

with pq.Program() as program:
    pq.Q() | pq.StateVector(occupation_numbers=(1, 1))

    pq.Q() | pq.Beamsplitter(theta=np.pi/4, phi=0.0)

    pq.Q() | pq.ParticleNumberMeasurement()

simulator = pq.SamplingSimulator(d=2)

result = simulator.execute(program, shots=shots)

counts = result.get_counts()

print("Relative frequency of (2, 0):", counts[(2, 0)] / shots)
print("Relative frequency of (0, 2):", counts[(0, 2)] / shots)

```

The outcome $(1,1)$ does not appear in the samples.

Exercise 1.7.4. From the previous exercise, derive the probabilities of detecting the outcomes $20, 11$ and 02! Verify that these correspond to the probabilities acquired by Piquasso simulations.

Solution. We can write

$$|1,1\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle , \tag{1.60}$$

and applying a 50/50 beamsplitter gate $\hat{U} = B_{12}(\theta = \pi/4, \phi = 0)$ we can write

$$\begin{aligned} \hat{U}|1,1\rangle &= \hat{U}\hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle \\ &= \hat{U}\hat{a}_1^\dagger \underbrace{\hat{U}^\dagger \hat{U}}_{=\mathbb{1}} \hat{a}_2^\dagger \underbrace{\hat{U}^\dagger \hat{U}}_{=\mathbb{1}} |0\rangle \\ &\overset{\hat{U}|0\rangle = |0\rangle}{=} \hat{U}\hat{a}_1^\dagger \hat{U}^\dagger \hat{U}\hat{a}_2^\dagger \hat{U}^\dagger |0\rangle . \end{aligned} \tag{1.61}$$

At this point, we can use the relation Eq. (1.31) to write

$$\hat{U}\hat{a}_1^\dagger \hat{U}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_2^\dagger), \tag{1.62}$$

$$\hat{U}\hat{a}_2^\dagger \hat{U}^\dagger = \frac{1}{\sqrt{2}}(-\hat{a}_1^\dagger + \hat{a}_2^\dagger). \tag{1.63}$$

which yields

$$\hat{U}|1,1\rangle = \frac{1}{2}(\hat{a}_1^\dagger + \hat{a}_2^\dagger)(-\hat{a}_1^\dagger + \hat{a}_2^\dagger)|0\rangle = \frac{1}{\sqrt{2}}(|2,0\rangle + |0,2\rangle), \tag{1.64}$$

where we have used the commutativity of the creation operators and Eq. 1.2. Hence, the probability of obtaining $(2,0)$ and $(0,2)$ is both 0.5, whereas it is not possible to obtain $(1,1)$. This effect is called the Hong-Ou-Mandel effect [37].

```
import numpy as np
import piquasso as pq

shots = 100

with pq.Program() as program:
    pq.Q() | pq.StateVector(occupation_numbers=(1, 1))

    pq.Q() | pq.Beamsplitter(theta=np.pi/4, phi=0.0)

```

```
simulator = pq.SamplingSimulator(d=2)

state = simulator.execute(program).state

print("Probability of (2, 0):", state.get_particle_detection_probability((2, 0)))
print("Probability of (0, 2):", state.get_particle_detection_probability((0, 2)))

```

Exercise 1.7.5. Consider the state $|4\rangle$, and apply photon loss to it with $\eta = 0.5$. What is the probability of detecting 2 photons in a subsequent photon detection? Is it possible that no photons remain?

Solution. It can be directly calculated by using Eq. (1.59):

$$\binom{n}{k}\eta^k(1-\eta)^{n-k} = \binom{4}{2} \times 0.5^2 \times 0.5^2 = 3/8. \tag{1.65}$$

Exercise 1.7.6. Model the previous setup using Piquasso using an environmental mode and a beamsplitter! Verify that the state vector corresponds to the expected state vector using Piquasso!

Solution. The Piquasso code is:

```
import piquasso as pq

with pq.Program() as program:
    pq.Q() | pq.StateVector(occupation_numbers=(4, 0))  # |1, 1>

    pq.Q(0, 1) | pq.Beamsplitter()

# Alternative syntax
# program = pq.Program(instructions=[pq.StateVector(occupation_numbers=(1, 1)).on_modes(0, 1)])

simulator = pq.SamplingSimulator(d=2)

result = simulator.execute(program, shots=20)

state = result.state

print("probability of |2, 2>: ", state.get_particle_detection_probability((2, 2)))

```

Chapter 2

Gaussian states

In this section, the concept of the Gaussian states[1] is introduced. Admitting many interesting properties, the set of Gaussian states is generally considered a tractable subset of all photonic quantum states. However, this chapter treats the already well-trodden path of Gaussian states only superficially. For more information regarding Gaussian states, the reader can visit Refs. [38, 33].

We start this chapter by introducing some elementary photonic states, which are the elementary examples of Gaussian states.

2.1 Elementary photonic quantum states

The vacuum state and Fock basis states have already been introduced in Section 1.1 in the context of creation and annihilation operators, but detailed definitions were omitted:

Definition 2.1.1 (Vacuum state)**.** The *vacuum state* is defined by

$$\hat{a}_i \left|0\right\rangle = 0, \qquad \forall\, i \in [d], \tag{2.1}$$

where the index i denotes the modes of the system.

Starting from vacuum as an initial state, the resulting states after applying a squeezing gate are called squeezed states:

Definition 2.1.2 (Squeezed state)**.** A single-mode *squeezed state* is defined by

$$\left|z\right\rangle \coloneqq S(z) \left|0\right\rangle, \tag{2.2}$$

where $z \in \mathbb{C}$.

Similarly, starting from vacuum as an initial state, the resulting states after applying a displacement gate are called coherent states:

Definition 2.1.3 (Coherent state)**.** A single-mode (canonical) *coherent state* is defined by

$$\left|\alpha\right\rangle \coloneqq D(\alpha) \left|0\right\rangle, \tag{2.3}$$

where $\alpha \in \mathbb{C}$.

Remark. Squeezed and coherent states have a very similar notation, but the difference is that coherent states are denoted by Greek letters (usually α), and squeezed states are denoted by Latin letters (usually z).

The vacuum, squeezed and coherent states are pure states. A common mixed state is the thermal state, which—instead of a state vector—is modeled by a *density operator*:

Definition 2.1.4 (Thermal state)**.** The density operator of a single-mode thermal state is defined as

$$\rho = \sum_{n=0}^{\infty} \frac{\overline{n}^n}{(1+\overline{n})^{n+1}} \left|n\right\rangle\!\left\langle n\right|, \tag{2.4}$$

where $\overline{n} = \mathrm{Tr}\left[\rho \hat{n}\right]$ is the average photon number.

[1] In this lecture notes, the term Gaussian states refer to *bosonic Gaussian states*.

The vacuum, coherent, squeezed and thermal states belong to a class of photonic quantum states called *Gaussian states*. However, with the exclusion of the vacuum state, the Fock basis states are *not* Gaussian states.

2.2 Linear gates and the symplectic group

Before introducing the concept of Gaussian states, it is useful to discuss the connection between linear gates and the symplectic group, as this provides a useful tool to analyze Gaussian states using their statistical moments.

As already discussed in Section 1.2, the action of linear optical gates on the ladder operators can help characterize the time evolution of photonic quantum states. More concretely, linear optical gates are special gates that act linearly on the ladder operators. This linear transformation can be characterized with the help of the real symplectic group [39]:

Definition 2.2.1 (Symplectic group)**.** The *real symplectic group* is defined as

$$\mathrm{Sp}(2d,\mathbb{R}) \coloneqq \left\{S \in \mathbb{R}^{2d\times 2d} \,|\, SJS^T = J\right\}, \tag{2.5}$$

where J is a non-singular skew-symmetric matrix, typically chosen as

$$J = \begin{bmatrix} & \mathbb{1} \\ -\mathbb{1} & \end{bmatrix}. \tag{2.6}$$

The symplectic group only describes linear optical gates with strictly quadratic Hamiltonians in the ladder operators. For general linear optical gates with Hamiltonians admitting both linear and quadratic terms, we need to introduce the real affine symplectic group.

Definition 2.2.2 (Affine symplectic group)**.** The *real affine symplectic group* is defined by the direct product[2]

$$\mathrm{ASp}(2d,\mathbb{R}) \coloneqq \mathbb{R}^{2d} \times \mathrm{Sp}(2d,\mathbb{R}), \tag{2.7}$$

where the group multiplication in $\mathrm{ASp}(2d,\mathbb{R})$ is defined by

$$(\boldsymbol{\zeta}_1, S_1)\cdot(\boldsymbol{\zeta}_2, S_2) \coloneqq (\boldsymbol{\zeta}_1 + S_1\boldsymbol{\zeta}_2, S_1S_2) \tag{2.8}$$

for $S_i \in \mathrm{Sp}(2d,\mathbb{R})$ and $\boldsymbol{\zeta}_i \in \mathbb{R}^{2d}$.

Remark. The real symplectic group is most often used in another, more convenient basis which we will sometimes call the *complex basis*. Hence, we define the *complex form* of the real symplectic group [40, 33]

$$\mathrm{Sp}^{(c)}(2d,\mathbb{R}) \coloneqq W\mathrm{Sp}(2d,\mathbb{R})W^\dagger = \left\{S \in \mathbb{C}^{2d\times 2d} : SKS^\dagger = K,\ W^\dagger SW \in \mathbb{R}^{2d\times 2d}\right\}, \tag{2.9}$$

where W is defined in Eq. (1.18) and

$$K = iWJW^\dagger = \begin{bmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{bmatrix}. \tag{2.10}$$

Similarly, the complex form of the affine symplectic group is just

$$\mathrm{ASp}^{(c)}(2d,\mathbb{R}) \coloneqq \mathrm{Sp}^{(c)}(2d,\mathbb{R}) \times W\mathbb{R}^{2d}, \tag{2.11}$$

where the group multiplication is defined analogously as in Definition 2.2.2.

The following proposition connects the CCR and the real symplectic group, i.e., the transformations which preserve the CCR is exactly the symplectic group from Defintion 2.2.1.

Proposition 1 (Transformations preserving CCR)**.** Consider a linear transformation on the ladder operators of the form

$$\boldsymbol{\omega} \coloneqq S\boldsymbol{\xi}, \tag{2.12}$$

where $S \in \mathbb{C}^{2d\times 2d}$ such that $W^\dagger SW \in \mathbb{R}^{2d\times 2d}$, and denote $\boldsymbol{\omega} = [\hat{b}_1, \ldots, \hat{b}_d, \hat{b}_1^\dagger, \ldots, \hat{b}_d^\dagger]^T$. If the transformed operators also fulfill the canonical commutation relations, i.e., when

$$[\hat{b}_i, \hat{b}_j^\dagger] = \delta_{ij}, \qquad [\hat{b}_i, \hat{b}_j] = [\hat{b}_i^\dagger, \hat{b}_j^\dagger] = 0, \tag{2.13}$$

then S must be a symplectic matrix in the complex basis, i.e.,

$$S \in \mathrm{Sp}^{(c)}(2d,\mathbb{R}). \tag{2.14}$$

[2] As a remark, this type of product between groups is called the semidirect product.

Proof. We can write $[\xi_i, \xi_j^\dagger] = [\omega_i, \omega_j^\dagger] = K_{ij}$, where K is defined in Eq. (2.10). Using this fact, the following is true:

$$[\omega_i, \omega_j^\dagger] = \sum_{k,l=1}^{2d} S_{ki}\overline{S_{lj}}[\xi_k, \xi_l^\dagger] = \sum_{k,l=1}^{2d} S_{ki}\overline{S_{lj}}K_{kl} \stackrel{!}{=} K_{ij}, \tag{2.15}$$

which implies that $SKS^\dagger = K$, hence $S \in \mathrm{Sp}^{(c)}(2d, \mathbb{R})$. □

A crucial observation is that evolving the ladder operators with linear optical quantum gates can be easily described, and the product of linear optical quantum gates, if it can be written using a single Hamiltonian by exponentiation, is also a linear optical quantum gate. In the following, we aim to show that linear optical quantum gates can be characterized by the affine symplectic group. For practical purposes, we describe gates admitting strictly linear and strictly quadratic Hamiltonians separately. Linear gates admitting strictly linear Hamiltonians can be described as follows:

Proposition 2 (Ladder operator time evolution with linear Hamiltonians)**.** Consider a linear optical gate described by the unitary operator

$$U = e^{i\hat{H}}, \tag{2.16}$$

where the Hamiltonian $\hat{H} = \breve{\boldsymbol{\alpha}} \cdot \boldsymbol{\xi}$ is strictly linear, following the notation from Definition 1.2.2. One can write the evolution of the ladder operators as

$$U^\dagger \boldsymbol{\xi} U = \boldsymbol{\xi} + iJ\breve{\boldsymbol{\alpha}}\mathbb{1}, \tag{2.17}$$

where we understand the last term as $iJ\breve{\boldsymbol{\alpha}}\mathbb{1} \coloneqq [(iJ\breve{\boldsymbol{\alpha}})_1\mathbb{1}, \ldots, (iJ\breve{\boldsymbol{\alpha}})_{2d}\mathbb{1}]^T$.

Proof. One may use the Hadamard lemma from Eq. (1.26) with the choice of $X = -i\hat{H}$ and $Y = \hat{a}_k$. The commutator can be expressed as

$$[\hat{H}, \hat{a}_k] = [\breve{\boldsymbol{\alpha}} \cdot \boldsymbol{\xi}, \hat{a}_k] = \sum_{j=1}^{d} \overline{\beta_j}[\hat{a}_j^\dagger, \hat{a}_k] = -\overline{\beta_k}\mathbb{1}, \tag{2.18a}$$

$$[\hat{H}, \hat{a}_k^\dagger] = -[\hat{H}, \hat{a}_k]^\dagger = \beta_k\mathbb{1}. \tag{2.18b}$$

Since the commutator is proportional to $\mathbb{1}$, we can omit the higher-order commutators and simply write

$$U^\dagger \hat{a}_k U = \hat{a}_k - \overline{\beta_k}\mathbb{1}, \tag{2.19a}$$

$$U^\dagger \hat{a}_k^\dagger U = \hat{a}_k^\dagger + \beta_k\mathbb{1}, \tag{2.19b}$$

or more succinctly,

$$U^\dagger \boldsymbol{\xi} U = \boldsymbol{\xi} + iJ\breve{\boldsymbol{\alpha}}\mathbb{1}. \tag{2.20}$$

□

For linear gates admitting strictly quadratic Hamiltonians, the adjoint action can be described by the complex form of the symplectic group [33]:

Proposition 3 (Ladder operator time evolution with quadratic Hamiltonians)**.** Consider a linear optical gate with the unitary operator $U = e^{i\hat{H}}$, where $\hat{H} = \boldsymbol{\xi}^\dagger H \boldsymbol{\xi}$ is a strictly quadratic Hamiltonian, following the notation from Definition 1.2.2. One can write the evolution of the ladder operators as

$$U^\dagger \boldsymbol{\xi} U = S^{(c)}\boldsymbol{\xi}, \tag{2.21}$$

where $S^{(c)} \in \mathrm{Sp}^{(c)}(2d, \mathbb{R})$ is given as

$$S^{(c)} = e^{i2KH}. \tag{2.22}$$

Proof. One may use the Hadamard lemma from Eq. (1.26), and write

$$U^\dagger \boldsymbol{\xi} U = e^{-i\hat{H}}\boldsymbol{\xi}e^{i\hat{H}} = e^{\mathrm{ad}_{(-i\hat{H})}}\boldsymbol{\xi}. \tag{2.23}$$

To determine $e^{\mathrm{ad}_{(-i\hat{H})}}$, one has to evaluate the action of $\mathrm{ad}_{(-i\hat{H})}$ on the ladder operators:

$$\mathrm{ad}_{(-i\hat{H})}\boldsymbol{\xi} = -i[\hat{H}, \boldsymbol{\xi}] = -i[\boldsymbol{\xi}^\dagger H\boldsymbol{\xi}, \boldsymbol{\xi}]. \tag{2.24}$$

According to Definition 1.2.2, H has the form

$$H = \begin{bmatrix} A & B \\ \overline{B} & \overline{A} \end{bmatrix}, \tag{2.25}$$

where $A^\dagger = A$ and $B^T = B$. Focusing on a single component, $[\boldsymbol{\xi}^\dagger H \boldsymbol{\xi}, \hat{a}_k]$ and $[\boldsymbol{\xi}^\dagger H \boldsymbol{\xi}, \hat{a}_k^\dagger]$ can be expressed as

$$\begin{aligned}
[\hat{H}, \hat{a}_k] &= \sum_{i,j=1}^{d} A_{ij}[\hat{a}_i^\dagger \hat{a}_j, \hat{a}_k] + \overline{A_{ij}}[\hat{a}_i \hat{a}_j^\dagger, \hat{a}_k] + B_{ij}[\hat{a}_i^\dagger \hat{a}_j^\dagger, \hat{a}_k] \\
&= -\sum_{i,j=1}^{d} A_{ij}\delta_{ik}\hat{a}_j + \overline{A_{ij}}\hat{a}_i\delta_{jk} + B_{ij}(\delta_{ik}\hat{a}_j^\dagger + \hat{a}_i^\dagger\delta_{ik}) \\
&= -\sum_{j=1}^{d} A_{kj}\hat{a}_j + B_{kj}\hat{a}_j^\dagger + \sum_{j=1}^{d} \underbrace{\overline{A_{jk}}}_{=A_{kj}}\hat{a}_j + \underbrace{B_{jk}}_{=B_{jk}}\hat{a}_j^\dagger \\
&= -2\sum_{j=1}^{d} A_{kj}\hat{a}_j + B_{kj}\hat{a}_j^\dagger,
\end{aligned} \tag{2.26a}$$

$$[\hat{H}, \hat{a}_k^\dagger] = -[\hat{H}, \hat{a}_k]^\dagger = 2\sum_{j=1}^{d} \overline{A_{kj}}\hat{a}_j^\dagger + \overline{B_{kj}}\hat{a}_j. \tag{2.26b}$$

By collecting all components the commutators can be organized as

$$[\boldsymbol{\xi}^\dagger H \boldsymbol{\xi}, \boldsymbol{\xi}] = -2KH\boldsymbol{\xi}. \tag{2.27}$$

Therefore, the transformation corresponding to the unitary evolution of the ladder operators can be written as

$$U^\dagger \boldsymbol{\xi} U = e^{i2KH}\boldsymbol{\xi}. \tag{2.28}$$

One can also show using Proposition 1, that $S^{(c)} \in \mathrm{Sp}^{(c)}(2d, \mathbb{R})$, since

$$[U^\dagger \boldsymbol{\xi} U, U^\dagger \boldsymbol{\xi}^\dagger U] = [\boldsymbol{\xi}, \boldsymbol{\xi}^\dagger], \tag{2.29}$$

and $W^\dagger S^{(c)} W \in \mathbb{R}^{2d\times 2d}$, since both $U^\dagger \hat{x}_i U$ and $U^\dagger \hat{p}_i U$ are self-adjoint for $i \in [d]$. □

Consequently, defining the action of the affine symplectic group $\mathrm{ASp}^{(c)}(2d, \mathbb{R})$ on the ladder operators $\boldsymbol{\xi}$ as

$$(\boldsymbol{\gamma}, S^{(c)}) \cdot \boldsymbol{\xi} \coloneqq S^{(c)}\boldsymbol{\xi} + \boldsymbol{\gamma}\mathbb{1}, \tag{2.30}$$

one can conclude that the evolution of the ladder operators by a linear optical circuit can be fully characterized by $\mathrm{ASp}^{(c)}(2d, \mathbb{R})$.

2.3 Gaussian states

As previously advertised, the set of Gaussian states is a subset of photonic quantum states having many particularly nice properties. For this, the main reason is that Gaussian states can be written using a single linear optical Hamiltonian as follows:

Definition 2.3.1 (Gaussian state)**.** Consider linear optical Hamiltonians $\hat{H}$ (called the *parent Hamiltonian*) for which the corresponding Hamiltonian matrix H, as defined in Definition 1.2.1, is positive definite. Denote the set of such operators by $\mathcal{A}$. The set of mixed Gaussian states is defined as

$$\mathcal{S}_{G,\mathrm{mixed}} \coloneqq \left\{ \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\, e^{-\beta\hat{H}}} : \hat{H} \in \mathcal{A}, \beta > 0 \right\}, \tag{2.31}$$

whereas the set of pure Gaussian states is defined as

$$\mathcal{S}_{G,\mathrm{pure}} \coloneqq \left\{ \lim_{\beta\to\infty} \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\, e^{-\beta\hat{H}}} : \hat{H} \in \mathcal{A} \right\}. \tag{2.32}$$

Hence, the set of Gaussian states is just $\mathcal{S}_G \coloneqq \mathcal{S}_{G,\mathrm{mixed}} \cup \mathcal{S}_{G,\mathrm{pure}}$.

We call Gaussian states that have no linear term in their parent Hamiltonian *centralized* Gaussian states. In this workshop, we focus on pure, centralied Gaussian states for brevity.

The vacuum state is the most important Gaussian state:

Example (Vacuum state). Let $\hat{H} = \hat{a}^\dagger \hat{a} = \hat{n} = \sum_{n=0}^{\infty} n \left|n\right\rangle\!\left\langle n\right|$, a linear optical Hamiltonian with positive definite Hamiltonian matrix. Then we can write

$$e^{-\beta \hat{H}} = \sum_{n=0}^{\infty} e^{-\beta n} \left|n\right\rangle\!\left\langle n\right|, \tag{2.33}$$

$$\operatorname{Tr} e^{-\beta \hat{H}} = \sum_{n=0}^{\infty} e^{-\beta n} = \frac{1}{1-e^{-\beta}}. \tag{2.34}$$

Then

$$\rho(\beta) = \frac{e^{-\beta \hat{H}}}{\operatorname{Tr} e^{-\beta \hat{H}}} = \sum_{n=0}^{\infty} (1-e^{-\beta}) e^{-\beta n} \left|n\right\rangle\!\left\langle n\right|, \tag{2.35}$$

and hence, when $\beta \to \infty$, we get

$$\lim_{\beta \to \infty} \rho(\beta) = \left|0\right\rangle\!\left\langle 0\right|, \tag{2.36}$$

hence, the vacuum state is a pure Gaussian state.

It should be remarked, that Eq. (2.35) defines a thermal state with mean photon number $\overline{n} = (1+e^{\beta})^{-1}$, and these are mixed Gaussian states for finite $\beta > 0$.

Gaussian states and linear optical gates are closely related since applying linear optical gates to Gaussian states yields Gaussian states. This can easily be verified by using the fact that linear optical Hamiltonians are mapped to linear optical Hamiltonians by applying linear optical gate unitaries. More concretely, for a mixed Gaussian state ρ, time evolution can be written as

$$U \rho U^\dagger = \frac{U e^{-\beta \hat{H}} U^\dagger}{\operatorname{Tr}\left[U e^{-\beta \hat{H}} U^\dagger\right]} = \frac{e^{-\beta U \hat{H} U^\dagger}}{\operatorname{Tr} e^{-\beta U \hat{H} U^\dagger}}, \tag{2.37}$$

and if U corresponds to a linear optical gate, then $U\hat{H}U^\dagger$ is also a linear optical Hamiltonian with a positive definite Hamiltonian matrix. The following is a prototypical example of single-mode pure Gaussian states:

Example (Squeezed state). A single-mode squeezed state defined by

$$\left|z\right\rangle \coloneqq S(z) \left|0\right\rangle, \tag{2.38}$$

is a Gaussian state, since the vacuum state is Gaussian, and linear gates map Gaussian states to Gaussian states.

2.4 Normal mode decomposition of Gaussian states

To explore the structure of Gaussian states, one first needs to be familiar with the concept of normal mode decomposition[3]:

Proposition 4 (Normal mode decomposition). Given a $2d \times 2d$ positive definite real matrix H, there exists a diagonal matrix D and a symplectic matrix $S \in \mathrm{Sp}(2d, \mathbb{R})$ such that

$$H = SDS^T, \tag{2.39}$$

where

$$D = \operatorname{diag}(\boldsymbol{\nu}, \boldsymbol{\nu}), \tag{2.40}$$

and $\nu_i > 0$ are the eigenvalues of $|iJH|$ and diag denotes the operation of forming a diagonal matrix with the prescribed vectors in the diagonal.

Proof. We omit the proof of this fact for brevity. For proof, the curious reader should visit Section 3.2.3 in Ref. [38]. □

[3] Also known as Williamson decomposition.

This can be used to decompose any quadratic Hamiltonian $\hat{H}$ from Eq. (1.21) admitting positive definite Hamiltonian matrix as

$$\hat{H} = \boldsymbol{\chi}^T H^{(\mathrm{xp})} \boldsymbol{\chi} = (S\boldsymbol{\chi})^T D S \boldsymbol{\chi} \tag{2.41}$$

for some real symplectic matrix $S \in \mathrm{Sp}(2d, \mathbb{R})$, and D has the form of Eq. (2.40). Now, take a linear gate with unitary operator U such that

$$U^\dagger \boldsymbol{\chi} U = S\boldsymbol{\chi}, \tag{2.42}$$

and hence

$$U\hat{H}U^\dagger = \boldsymbol{\chi}^T D \boldsymbol{\chi} = \sum_{i=1}^{d} \nu_i (\hat{x}_i^2 + \hat{p}_i^2) = \sum_{i=1}^{d} 2\nu_i (\hat{n}_i + \frac{1}{2}\mathbb{1}). \tag{2.43}$$

Thus, every quadratic Hamiltonian with positive definite Hamiltonian matrix is unitarily equivalent to a Hamiltonian of free, non-interacting harmonic oscillators with frequencies proportional to ν_i.

2.5 Statistical moments of Gaussian states

The direct consequence of the normal mode decomposition of quadratic Hamiltonians is that (pure) Gaussian states can only be produced by evolving vacuum states solely with linear optical Hamiltonians. Hence, Gaussian states can be fully characterized by their displacement vector μ and covariance matrix σ defined elementwise by

$$\begin{aligned} \mu_i &\coloneqq \mathrm{Tr}\left[\rho\, \boldsymbol{\chi}_i\right], \\ \sigma_{ij} &\coloneqq \mathrm{Tr}\left[\rho\, \{\boldsymbol{\chi}_i - \mu_i \mathbb{1}, \boldsymbol{\chi}_j - \mu_j \mathbb{1}\}\right], \end{aligned} \tag{2.44}$$

where $\{A, B\} \coloneqq AB + BA$ is the anticommutator and $\boldsymbol{\chi}$ is a vector of quadrature operators from in Eq. (1.18) from Section 1.1. However, sometimes it is more practical to use the analog quantities in the ladder operator basis:

$$\mu_i^{(\mathrm{c})} \coloneqq \mathrm{Tr}\left[\rho\, \boldsymbol{\xi}_i\right], \tag{2.45}$$

$$\sigma_{ij}^{(\mathrm{c})} \coloneqq \mathrm{Tr}\left[\rho\, \{(\boldsymbol{\xi}_i - \mu_i^{(\mathrm{c})}\mathbb{1})^\dagger, \boldsymbol{\xi}_j - \mu_j^{(\mathrm{c})}\mathbb{1}\}\right]. \tag{2.46}$$

We can transition from the two definitions as

$$\boldsymbol{\mu}^{(\mathrm{c})} \coloneqq W\boldsymbol{\mu}, \tag{2.47}$$

$$\sigma^{(\mathrm{c})} \coloneqq W\sigma W^\dagger, \tag{2.48}$$

where W is defined in Eq. (1.18).

To gain some intuition, we consider the following example:

Example (Covariance matrix of the vacuum state). We know that if $\rho = |0\rangle\langle 0|$, then

$$\mu_0^{(\mathrm{c})} = \mathrm{Tr}\left[|0\rangle\langle 0|\, a\right] = \langle 0|\, \underbrace{a\,|0\rangle}_{=0} = 0, \tag{2.49}$$

$$\mu_1^{(\mathrm{c})} = \mathrm{Tr}\left[|0\rangle\langle 0|\, \hat{a}^\dagger\right] = \langle 0|\, \underbrace{\hat{a}^\dagger\,|0\rangle}_{=|1\rangle} = \langle 0|1\rangle = 0. \tag{2.50}$$

Then the elements of the covariance matrix can be written as

$$\sigma_{ij}^{(\mathrm{c})} = \mathrm{Tr}\left[\rho\{\boldsymbol{\xi}_i^\dagger, \boldsymbol{\xi}_j\}\right] = \langle 0|\, \{\boldsymbol{\xi}_i^\dagger, \boldsymbol{\xi}_j\}\, |0\rangle\,. \tag{2.51}$$

Analogously to the previous argument, we know that $\langle 0|\, \hat{a}^\dagger \hat{a}^\dagger\, |0\rangle = \langle 0|\, aa\, |0\rangle = 0$, and that $\{\hat{a}^\dagger, a\} = 2\hat{a}^\dagger a + \mathbb{1}$, meaning that the statistical moments of the vacuum state are simply

$$\boldsymbol{\mu}^{(\mathrm{c})} = [0, 0]^T, \tag{2.52}$$

$$\sigma^{(\mathrm{c})} = \mathbb{1}_{2\times 2}, \tag{2.53}$$

which—incidentally—has the same form in the quadrature basis.

The evolution by a linear gate with admitting a strictly quadratic Hamiltonian can be fully characterized by the following transformation of the displacement vector and covariance matrix as

$$\begin{aligned} \boldsymbol{\mu} &\mapsto S\boldsymbol{\mu}, \\ \sigma &\mapsto S\sigma S^T, \end{aligned} \tag{2.54}$$

where $S \in \mathrm{Sp}(2d, \mathbb{R})$ is a real symplectic matrix corresponding to the linear gate, as discussed in Section 2.2.

2.6 Bloch-Messiah decomposition

As in the case of the Clements decomposition from Section 1.3.2, one might naturally be led to the question of decomposing general linear optical elements. The answer essentially boils down to a certain decomposition of real symplectic matrices, an indispensable tool for studying linear optical quantum computing.

It turns out, that only displacement, squeezing, phaseshifter, and beamsplitter gates are sufficient for implementing any linear operation in photonic quantum computing. The tool for acquiring the decomposition into elementary linear optical gates is called *Bloch-Messiah decomposition*[4]:

Theorem 2.6.1 (Bloch-Messiah decomposition)**.** Consider a symplectic matrix $S^{(c)} \in \mathrm{Sp}^{(c)}(2d, \mathbb{R})$. One can decompose $S^{(c)}$ as

$$S^{(c)} = V_1 D V_2, \qquad V_1 = \begin{bmatrix} U_1 & \\ & \overline{U}_1 \end{bmatrix} \qquad V_2 = \begin{bmatrix} U_2 & \\ & \overline{U}_2 \end{bmatrix}, \tag{2.55}$$

where $U_1, U_2 \in \mathrm{U}(d)$ and D is

$$D = \begin{bmatrix} \cosh \boldsymbol{r} & -\sinh \boldsymbol{r} \\ -\sinh \boldsymbol{r} & \cosh \boldsymbol{r} \end{bmatrix}, \tag{2.56}$$

and $\boldsymbol{r} \in \mathbb{R}^d_{\geq 0}$. Moreover, cosh and sinh are understood element-wise, e.g., $\cosh \boldsymbol{r} = \bigoplus_{i=1}^d \cosh r_i$.

Proof. We omit the proof of this linear algebraic fact for brevity, however, a proof can be found in Ref. [41]. □

Using this decomposition, any unitary operator corresponding to a quadratic Hamiltonian can be decomposed into two passive linear transformations (which can be further decomposed via the Clements decomposition from Section 1.3.2) and a series of single-mode squeezing gates (by using Eq. (1.35)). Linear optical gates also contain linear terms in the Hamiltonian, which yield a column of single-mode displacement gates. In the following, we will organize these thoughts in a theorem:

Theorem 2.6.2. The unitary operator U of any linear optical gate can be decomposed as

$$U = \left(\prod_{i=1}^d D_i(\alpha_i) \right) I(U_1) \left(\prod_{i=1}^d S_i(r_i) \right) I(U_2), \qquad \alpha_i \in \mathbb{C},\, r_i \in \mathbb{R}, \tag{2.57}$$

where I_1, I_2 are interferometers parametrized by U_1 and U_2 unitaries, $D_i(\alpha_i)$ and $S_i(r_i)$ are the displacement, and squeezing gates acting on the i-th mode.

Proof. As discussed in Section 2.2, we know that any linear optical gate determines an element of the affine symplectic group $\mathrm{ASp}^{(c)}(2d, \mathbb{R})$. According to Eq. (2.8), we can decompose any $(\boldsymbol{\gamma}, S^{(c)}) \in \mathrm{ASp}^{(c)}(2d, \mathbb{R})$ simply as

$$(\boldsymbol{\gamma}, S^{(c)}) = (\boldsymbol{\gamma}, \mathbb{1}) \cdot (\boldsymbol{0}, S^{(c)}), \tag{2.58}$$

which correspond to linear and quadratic terms in the Hamiltonian, respectively. Furthermore, applying Bloch-Messiah decomposition from Theorem 2.6.1 to the symplectic matrix $S^{(c)}$ we get

$$S^{(c)} = V_1 D V_2, \tag{2.59}$$

where V_1, V_2 are unitary symplectic matrices corresponding to interferometers $I(U_1)$ and $I(U_2)$, and D corresponds to a column of squeezing gates as

$$S(\boldsymbol{r}) \coloneqq \prod_{i=1}^d S(r_i). \tag{2.60}$$

Finally, the linear part $\boldsymbol{\gamma}$ is represented on the bosonic Fock space with a column of single-mode displacement gates. □

[4] Also called *Euler* decomposition in the literature.

2.7 Exercises

Exercise 2.7.1. Calculate the probability of detecting 0 photons in a coherent state, i.e., calculate

$$|\langle 0|\alpha\rangle|^2 = ? \tag{2.61}$$

Use the formula

$$D(\alpha) = e^{\alpha\hat{a}^\dagger - \overline{\alpha}a} = e^{\alpha\hat{a}^\dagger} e^{-\overline{\alpha}a} e^{\frac{1}{2}[\alpha\hat{a}^\dagger, \overline{\alpha}a]}. \tag{2.62}$$

Solution. We can evaluate $[\alpha\hat{a}^\dagger, \overline{\alpha}a] = -|\alpha|^2$ and write

$$\langle 0|\alpha\rangle = \langle 0|\, D(\alpha)\, |0\rangle = e^{-\frac{|\alpha|^2}{2}} \langle 0|\, e^{\alpha\hat{a}^\dagger} e^{-\overline{\alpha}a}\, |0\rangle \overset{a|0\rangle=0}{=} e^{-\frac{|\alpha|^2}{2}} \underbrace{\langle 0|0\rangle}_{=1} = e^{-\frac{|\alpha|^2}{2}}. \tag{2.63}$$

This yields $|\langle 0|\alpha\rangle|^2 = e^{-|\alpha|^2}$. We can also verify this in Piquasso:

```
import piquasso as pq
import numpy as np


alpha = 0.1

with pq.Program() as program:
    pq.Q() | pq.Vacuum()

    pq.Q() | pq.Displacement(alpha)

simulator = pq.PureFockSimulator(d=1)

state = simulator.execute(program).state

print("Expected: ", np.exp(-np.abs(alpha)**2))
print("Actual: ", state.get_particle_detection_probability((0,)))

```

Exercise 2.7.2. Use Piquasso's `PureFockSimulator` to prepare coherent states using the `Displacement` instruction! Why is the norm of the simulated state less than 1? Verify that the norm increases by increasing the `cutoff` in `Config`!

Solution. The Piquasso code is:

```
import piquasso as pq
import numpy as np


with pq.Program() as program:
    pq.Q() | pq.Vacuum()

    pq.Q() | pq.Displacement(r=0.1)

simulator = pq.PureFockSimulator(d=1, config=pq.Config(cutoff=4)) # <- Increase cutoff!

state = simulator.execute(program).state

print("Norm: ", state.norm)

```

Exercise 2.7.3. What is the average number of particles in a coherent state, i.e., what is

$$\langle \hat{n} \rangle_{D(\alpha)|0\rangle} = \langle 0|\, D^\dagger(\alpha) \hat{n} D(\alpha)\, |0\rangle ? \tag{2.64}$$

Compare the results with results from `PureFockSimulator` and `GaussianSimulator`!

Solution. The Piquasso code is:

```
import piquasso as pq
import numpy as np


with pq.Program() as program:
    pq.Q() | pq.Vacuum()

    pq.Q() | pq.Displacement(r=0.1)

simulator = pq.PureFockSimulator(d=1) # Change PureFockSimulator -> GaussianSimulator!

state = simulator.execute(program).state

print("Mean photon number: ", state.mean_photon_number())

```

Exercise 2.7.4. Calculate the covariance matrix of a single-mode coherent state! Use `GaussianSimulator` to compare the results!

Solution. Let us calculate $\sigma^{(c)}$. But first, we have to start with $\boldsymbol{\mu}^{(c)}$, which can be calculated easily from Eq. (1.38):

$$\boldsymbol{\mu}^{(c)} = \breve{\alpha}, \tag{2.65}$$

where $\breve{\alpha} \coloneqq [\alpha, \overline{\alpha}]^T$. Here, we used that $|\alpha\rangle = D(\alpha)|0\rangle$ and we used the cyclicity of the trace to rewrite the expression in the Heisenberg picture. For the covariance matrix, we can also write the evolution of the anticommutator as

$$\begin{aligned} D^\dagger(\alpha)\{\xi_i^\dagger - \overline{\breve{\alpha}_i}\mathbb{1}, \xi_j - \breve{\alpha}_j\mathbb{1}\}D(\alpha) &= \{D^\dagger(\alpha)\xi_i^\dagger D(\alpha) - \overline{\breve{\alpha}_i}\mathbb{1}, D^\dagger(\alpha)\xi_j D(\alpha) - \breve{\alpha}_j\mathbb{1}\} \\ &= \{\xi_i^\dagger + \overline{\breve{\alpha}_i}\mathbb{1} - \overline{\breve{\alpha}_i}\mathbb{1}, \xi_j + \breve{\alpha}_j\mathbb{1} - \breve{\alpha}_j\mathbb{1}\} \\ &= \{\xi_i^\dagger, \xi_j\}, \end{aligned} \tag{2.66}$$

which will yield the covariance matrix of the vacuum, which is $\mathbb{1}_{2\times 2}$. Hence $\sigma^{(c)} = \mathbb{1}_{2\times 2}$. We can verify this in Piquasso:

```
import piquasso as pq
import numpy as np


with pq.Program() as program:
    pq.Q() | pq.Vacuum()

    pq.Q() | pq.Displacement(r=0.1)

simulator = pq.GaussianSimulator(d=1)

state = simulator.execute(program).state

print("Covariance matrix: ", state.complex_covariance_matrix)

```

Exercise 2.7.5. What is the average number of particles in a squeezed state, i.e., what is

$$\langle \hat{n} \rangle_{S(z)|0\rangle} = \langle 0| S^\dagger(z) \hat{n} S(z) |0\rangle ? \tag{2.67}$$

Compare the results with results from `PureFockSimualator` and `GaussianSimulator`!

Solution. We know that

$$S^\dagger(z)\hat{n}S(z) = S^\dagger(z)\hat{a}^\dagger\hat{a}S(z) = S^\dagger(z)\hat{a}^\dagger S(z) S^\dagger(z)\hat{a}S(z), \tag{2.68}$$

and we can write that

$$\begin{aligned} S^\dagger(z)\hat{a}^\dagger S(z) &= \cosh(r)\hat{a}^\dagger - e^{-i\phi}\sinh(r)\hat{a}, \\ S^\dagger(z)\hat{a}S(z) &= \cosh(r)\hat{a} - e^{i\phi}\sinh(r)\hat{a}^\dagger, \end{aligned} \tag{2.69}$$

where $z = re^{i\phi}$. Hence, we can write

$$S^\dagger(z)\hat{n}S(z) = \cosh^2(r)\hat{a}^\dagger\hat{a} - \cosh(r)\sinh(r)(e^{i\phi}(\hat{a}^\dagger)^2 + e^{-i\phi}\hat{a}\hat{a})\sinh^2(r)\hat{a}\hat{a}^\dagger. \tag{2.70}$$

Hence,

$$\langle z|\, \hat{n}\, |z\rangle = \langle 0| S^\dagger(z)\hat{n}S(z)|0\rangle = \sinh^2(r). \tag{2.71}$$

Exercise 2.7.6. Show that the covariance matrix of a single-mode squeezed state, has the form

$$\sigma = \begin{bmatrix} e^{-2r} & 0 \\ 0 & e^{2r}, \end{bmatrix} \tag{2.72}$$

or, in the ladder operator basis,

$$\sigma^{(c)} = \begin{bmatrix} \cosh(2r) & -e^{i\phi}\sinh(2r) \\ -e^{-i\phi}\sinh(2r) & \cosh(2r) \end{bmatrix}. \tag{2.73}$$

Use `GaussianSimulator` to compare the results!

Solution. Since the covariance matrix of the vacuum is $\mathbb{1}_{2\times 2}$, using Eq. (2.54) we can write that $\sigma = SS^T$, where S is the (real) symplectic matrix of the squeezing gate from Eq. (1.36).

Chapter 3

Phase space formulation of quantum optics

This chapter provides a quick and dirty introduction to the phase space formulation of quantum optics, a very useful formalism. For this, we will use the following handy formula for the Gaussian integrals:

$$\int_{\mathbb{C}^{2d}} \exp\left(-\frac{1}{2}\breve{\boldsymbol{\alpha}}^\dagger \Sigma \breve{\boldsymbol{\alpha}} + \breve{\boldsymbol{\beta}}^\dagger \breve{\boldsymbol{\alpha}}\right) \mathrm{d}^{2d}\boldsymbol{\alpha} = \frac{(2\pi)^d}{\sqrt{\det \Sigma}} \exp\left(\frac{1}{2}\breve{\boldsymbol{\beta}}^\dagger \Sigma^{-1} \breve{\boldsymbol{\beta}}\right), \tag{3.1}$$

for a self-adjoint $\Sigma \in \mathbb{C}^{2d\times 2d}$ such that $W^\dagger \Sigma W \in \mathbb{R}^{2d\times 2d}$ and $\boldsymbol{\beta} \in \mathbb{C}^d$

3.1 Weyl operators and coherent states

An important property of coherent states from Section 2.1 is that coherent states are eigenvectors of the annihilation operator, i.e.,

$$a\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{3.2}$$

since

$$a\,|\alpha\rangle = D(\alpha)D^\dagger(\alpha) a D(\alpha)\,|0\rangle = D(\alpha)(a+\alpha)\,|0\rangle = \alpha D(\alpha)\,|0\rangle = \alpha\,|\alpha\rangle\,. \tag{3.3}$$

Using this, we can derive

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle\,. \tag{3.4}$$

Moreover, using the Baker-Campbell-Hausdorff formula, we can write

$$D(\alpha) = e^{\alpha\hat{a}^\dagger - \overline{\alpha}a} = e^{\alpha\hat{a}^\dagger} e^{-\overline{\alpha}a} e^{\frac{1}{2}[\alpha\hat{a}^\dagger, \overline{\alpha}a]} = e^{\alpha\hat{a}^\dagger} e^{-\overline{\alpha}a} e^{-\frac{|\alpha|^2}{2}}. \tag{3.5}$$

The Weyl operators admit the following important property:

Proposition 5. The Weyl operators admit the following multiplicative property

$$D(\alpha)D(\beta) = e^{\frac{1}{2}(\alpha\overline{\beta} - \overline{\alpha}\beta)} D(\alpha+\beta). \tag{3.6}$$

Proof. We use the special case of the Baker-Campbell-Hausdorff formula, which states that

$$e^X e^Y = e^{X+Y} e^{\frac{1}{2}[X,Y]} \tag{3.7}$$

when $[X,Y]$ commutes with both X and Y. In our case, we have

$$X = \alpha\hat{a}^\dagger - \overline{\alpha}a, \tag{3.8a}$$

$$Y = \beta\hat{a}^\dagger - \overline{\beta}a, \tag{3.8b}$$

and indeed, we can write

$$[X,Y] = (\alpha\overline{\beta} - \overline{\alpha}\beta)\underbrace{[a,\hat{a}^\dagger]}_{=\mathbb{1}} = (\alpha\overline{\beta} - \overline{\alpha}\beta)\mathbb{1}, \tag{3.9}$$

from which we can immediately conclude

$$D(\alpha)D(\beta) = e^{\frac{1}{2}(\alpha\overline{\beta}-\overline{\alpha}\beta)}D(\alpha+\beta). \tag{3.10}$$

□

A corollary of this is that the Weyl operators commute up to a phase, i.e.,

$$D(\alpha)D(\beta) = e^{\alpha\overline{\beta}-\overline{\alpha}\beta}D(\beta)D(\alpha) \tag{3.11}$$

As another corollary, we can write that the overlap of coherent states is

$$\langle\beta|\alpha\rangle = \langle 0|\, D(-\beta)D(\alpha)\,|0\rangle = e^{-\frac{1}{2}(\alpha\overline{\beta}-\overline{\alpha}\beta)}\underbrace{\langle 0|\, D(\alpha-\beta)\,|0\rangle}_{=\langle 0|\alpha-\beta\rangle} = e^{-\frac{1}{2}(\alpha\overline{\beta}-\overline{\alpha}\beta)}e^{-\frac{1}{2}|\alpha-\beta|^2} \neq \delta^{(2)}(\alpha-\beta), \tag{3.12}$$

where $\delta^{(2)}(\gamma)$ is understood as $\delta^{(2)}(\gamma) = \delta(\mathrm{Re}(\gamma))\delta(\mathrm{Im}(\gamma))$ and δ is the Dirac delta distribution. This means, that (unsurprisingly) the coherent states are not orthogonal. However, we can show that coherent state form a *complete set*, i.e., the identity operator can be written as a weighted integral of coherent states.

Proposition 6 (Resolution of identity)**.** The coherent states resolve the identity operator on the Fock space, i.e.,

$$\mathbb{1} = \frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha, \tag{3.13}$$

where $\mathrm{d}^2\alpha$ is a shorthand for $\mathrm{d}\,\mathrm{Re}\,\alpha\,\mathrm{d}\,\mathrm{Im}\,\alpha$.

Proof. Insert Eq. (3.4) and switch to polar coordinates. Write directly

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha = \sum_{n,m=0}^{\infty}\frac{1}{\pi\sqrt{n!m!}}\int_{\mathbb{C}} e^{-|\alpha|^2}\alpha^n\overline{\alpha}^m\mathrm{d}^2\alpha\,|n\rangle\langle m|\,. \tag{3.14}$$

We can write the integral in polar coordinates as

$$\int_{\mathbb{C}} e^{-|\alpha|^2}\alpha^n\overline{\alpha}^m\mathrm{d}^2\alpha = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}r^{n+m+1}e^{i(n-m)\varphi}\mathrm{d}r\mathrm{d}\varphi = 2\pi\delta_{nm}\underbrace{\int_0^{\infty} e^{-r^2}r^{2n+1}\mathrm{d}r}_{=\frac{n!}{2}} = \pi\delta_{nm}n!, \tag{3.15}$$

and by directly substituting we get

$$\frac{1}{\pi}\int_{\mathbb{C}} |\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha = \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{1}. \tag{3.16}$$

□

Since the coherent states resolve the identity but are *not* orthogal, they form an *overcomplete set*, i.e., a set which is still complete after removal of any element. Hence, the trace of an operator $\hat{O}$ can be written as an integral over the phase space:

$$\mathrm{Tr}\,\hat{O} = \sum_{n=0}^{\infty}\langle n|\,\hat{O}\,|n\rangle = \frac{1}{\pi}\int_{\mathbb{C}}\langle\alpha|\,\hat{O}\,|\alpha\rangle\,\mathrm{d}^2\alpha. \tag{3.17}$$

To generalize this for multiple qumodes, a d-mode Weyl operator can be written as

$$D(\boldsymbol{\alpha}) \coloneqq \prod_{i=1}^{d} D(\alpha_i) = \exp\left(-\breve{\boldsymbol{\alpha}}^\dagger K\boldsymbol{\xi}\right), \tag{3.18}$$

where $\breve{\boldsymbol{\alpha}} \coloneqq [\boldsymbol{\alpha}, \overline{\boldsymbol{\alpha}}]^T$ and, as before,

$$K \coloneqq \begin{bmatrix}\mathbb{1} & \\ & -\mathbb{1}\end{bmatrix}. \tag{3.19}$$

where $\boldsymbol{\xi}$ is the vector of ladder operators defined in Section 1.1. A very useful relation for the multimode Weyl operator is as follows. Consider a linear gate U and its complex form symplectic matrix S. Then

$$U^\dagger D(\boldsymbol{\alpha})U = \exp\left(-\breve{\boldsymbol{\alpha}}^\dagger KS\boldsymbol{\xi}\right) = \exp\left(-\breve{\boldsymbol{\alpha}}^\dagger(S^{-1})^\dagger K\boldsymbol{\xi}\right) = \exp\left(-(S^{-1}\breve{\boldsymbol{\alpha}})^\dagger K\boldsymbol{\xi}\right) = D((S^{-1}\breve{\boldsymbol{\alpha}})_{1:d}). \tag{3.20}$$

This means, that evolving the Weyl operator by a Gaussian gate amounts to a basis change in the phase space.

3.2 Fourier-Weyl relation

We can now establish a connection between operators and functions over the phase space, which relies on the fact that the Weyl operators form an orthogonal complete set on the space of operators with respect to the Hilbert-Schmidt inner product defined by $\langle A, B\rangle_{\mathrm{HS}} \coloneqq \operatorname{Tr} A^\dagger B$.

Proposition 7. The trace of the Weyl operator can be calculated as follows:

$$\operatorname{Tr} D(\alpha) = \pi\delta^{(2)}(\alpha), \tag{3.21}$$

where $\delta^{(2)}(\alpha)$ is understood as $\delta^{(2)}(\alpha) = \delta(\mathrm{Re}(\alpha))\delta(\mathrm{Im}(\alpha))$ and δ is the Dirac delta distribution.

Proof. We can write

$$\operatorname{Tr} D(\alpha) = \frac{1}{\pi}\int_{\mathbb{C}} \langle\beta| D(\alpha) |\beta\rangle \,\mathrm{d}^2\beta \tag{3.22}$$

The Weyl operator can be decomposed as

$$D(\alpha) = e^{\alpha\hat{a}^\dagger} e^{-\overline{\alpha}a} e^{-\frac{|\alpha|^2}{2}}, \tag{3.23}$$

and therefore

$$\langle\beta| D(\alpha) |\beta\rangle = e^{\alpha\overline{\beta}} e^{-\overline{\alpha}\beta} e^{-\frac{|\alpha|^2}{2}}, \tag{3.24}$$

and hence

$$\operatorname{Tr} D(\alpha) = e^{-\frac{|\alpha|^2}{2}} \frac{1}{\pi}\int_{\mathbb{C}} e^{\alpha\overline{\beta}} e^{-\overline{\alpha}\beta} \mathrm{d}^2\beta. \tag{3.25}$$

Now write $\alpha = a + ib$ and $\beta = x + iy$, so that we can write

$$\alpha\overline{\beta} - \overline{\alpha}\beta = 2i(bx - ay), \tag{3.26}$$

and hence

$$\operatorname{Tr} D(\alpha) = e^{-\frac{|\alpha|^2}{2}} \frac{1}{\pi}\int_{-\infty}^{\infty} e^{2ibx}\mathrm{d}x \int_{-\infty}^{\infty} e^{-2iay}\mathrm{d}y. \tag{3.27}$$

Finally, we use the relation

$$\delta(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iax}\mathrm{d}x \tag{3.28}$$

to obtain

$$\operatorname{Tr} D(\alpha) = e^{-\frac{|\alpha|^2}{2}} \frac{1}{\pi}\pi^2\delta(a)\delta(b) = \pi\delta(a)\delta(b) = \pi\delta^{(2)}(\alpha). \tag{3.29}$$

□

Proposition 8. The vacuum state can be represented as a weighted integral over the Weyl operators as

$$|0\rangle\langle 0| = \frac{1}{\pi}\int_{\mathbb{C}} e^{-\frac{|\gamma|^2}{2}} D(\gamma)\mathrm{d}^2\gamma. \tag{3.30}$$

Proof. Let us denote

$$\hat{O} = \frac{1}{\pi}\int_{\mathbb{C}} e^{-\frac{|\gamma|^2}{2}} D(\gamma)\mathrm{d}^2\gamma. \tag{3.31}$$

We can immediately see that this operator is an orthogonal projector, i.e., $\hat{O}^2 = \hat{O}^\dagger = \hat{O}$. Moreover, it has unit trace, since $\operatorname{Tr} D(\alpha) = \pi\delta^{(2)}(\alpha)$. Finally, we know that it fully overlaps with $|0\rangle\langle 0|$:

$$\langle 0|\hat{O}|0\rangle = \frac{1}{\pi}\int_{\mathbb{C}} e^{-|\gamma|^2}\mathrm{d}^2\gamma = 1. \tag{3.32}$$

Therefore, it must be that $\hat{O} = |0\rangle\langle 0|$. □

This means, that we can write the vacuum as a weighted integral over Weyl operators. This fact is true more generally, and Proposition 8 is only a special case of the following relation:

Theorem 3.2.1 (Fourier-Weyl relation)**.** Given a bounded operator $\hat{O}$ on the single-mode bosonic Fock space, we can write it as a weighted integral over the phase space as

$$\hat{O} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{Tr}\left[D(\alpha)\hat{O}\right] D(-\alpha)\mathrm{d}^2\alpha. \tag{3.33}$$

Proof. Firstly, any bounded operator $\hat{O}$ can be written as

$$\hat{O} = \frac{1}{\pi^2}\int_{\mathbb{C}^2} \langle\alpha|\,\hat{O}\,|\beta\rangle\,|\alpha\rangle\langle\beta|\,\mathrm{d}^2\alpha\mathrm{d}^2\beta, \tag{3.34}$$

Therefore, if we show that

$$|\alpha\rangle\langle\beta| = \frac{1}{\pi}\int_{\mathbb{C}} \langle\beta|\,D(\gamma)\,|\alpha\rangle\,D(-\gamma)\mathrm{d}^2\gamma, \tag{3.35}$$

we are done. We can calculate directly

$$D(-\beta)D(\gamma)D(\alpha) \overset{\text{Proposition 5}}{=} e^{-\frac{1}{2}(\beta\overline{\gamma}-\overline{\beta}\gamma)}D(\gamma-\beta)D(\alpha) \overset{\text{Proposition 5}}{=} e^{-\frac{1}{2}(\beta\overline{\gamma}-\overline{\beta}\gamma)}e^{\frac{1}{2}((\gamma-\beta)\overline{\alpha}-\overline{(\gamma-\beta)}\alpha)}D(\alpha+\gamma-\beta), \tag{3.36}$$

which yields

$$\langle\beta|\,D(\gamma)\,|\alpha\rangle = e^{-\frac{1}{2}(\beta\overline{\gamma}-\overline{\beta}\gamma)}e^{\frac{1}{2}((\gamma-\beta)\overline{\alpha}-\overline{(\gamma-\beta)}\alpha)}e^{-\frac{|\alpha+\gamma-\beta|^2}{2}} = \exp\Big(-\tfrac{1}{2}(|\alpha|^2+|\gamma|^2+|\beta|^2)\;+\;\alpha\bar{\beta}\;-\;\alpha\bar{\gamma}\;+\;\gamma\bar{\beta}\Big). \tag{3.37}$$

Using Proposition 8, we can write

$$|\alpha\rangle\langle\beta| = D(\alpha)\,|0\rangle\langle 0|\,D(-\beta) = \frac{1}{\pi}\int_{\mathbb{C}} e^{-\frac{|\gamma|^2}{2}}D(\alpha)D(\gamma)D(-\beta)\mathrm{d}^2\gamma, \tag{3.38}$$

where we can write

$$D(\alpha)D(\gamma)D(-\beta) \overset{\text{Proposition 5}}{=} e^{\frac{1}{2}(\alpha\overline{\gamma}-\overline{\alpha}\gamma)}D(\alpha+\gamma)D(-\beta) \overset{\text{Proposition 5}}{=} e^{-\frac{1}{2}(\alpha\overline{\beta}-\overline{\alpha}\beta)}e^{\frac{1}{2}((\alpha+\beta)\overline{\gamma}-\overline{(\alpha+\beta)}\gamma)}D(\alpha+\gamma-\beta), \tag{3.39}$$

which means that

$$\begin{aligned}
|\alpha\rangle\langle\beta| &= \frac{1}{\pi}e^{-\frac{1}{2}(\alpha\overline{\beta}-\overline{\alpha}\beta)}\int_{\mathbb{C}} e^{-\frac{|\gamma|^2}{2}}e^{\frac{1}{2}((\alpha+\beta)\overline{\gamma}-\overline{(\alpha+\beta)}\gamma)}D(\alpha+\gamma-\beta)\mathrm{d}^2\gamma \\
&\overset{-\sigma=\alpha+\gamma-\beta}{=} \frac{1}{\pi}e^{-\frac{1}{2}(\alpha\overline{\beta}-\overline{\alpha}\beta)}\int_{\mathbb{C}} e^{-\frac{|\beta-\alpha-\sigma|^2}{2}}e^{\frac{1}{2}((\alpha+\beta)\overline{(\beta-\alpha-\sigma)}-\overline{(\alpha+\beta)}(\beta-\alpha-\sigma))}D(-\sigma)\mathrm{d}^2\sigma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \underbrace{\exp\Big(-\tfrac{1}{2}\big(|\alpha|^2+|\beta|^2+|\sigma|^2\big)\;+\;\alpha\overline{\beta}\;-\;\alpha\overline{\sigma}\;+\;\sigma\overline{\beta}\Big)}_{=\langle\beta|D(\sigma)|\alpha\rangle} D(-\sigma)\mathrm{d}^2\sigma \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \langle\beta|\,D(\sigma)\,|\alpha\rangle\,D(-\sigma)\mathrm{d}^2\sigma.
\end{aligned} \tag{3.40}$$

□

3.3 Characteristic functions and Wigner function

The Fourier-Weyl relation is the main motivation for defining the following function:

Definition 3.3.1 (Characteristic function)**.** Given a state ρ, we define the *characteristic function* as

$$\chi_\rho(\alpha) \coloneqq \mathrm{Tr}\left[D(\alpha)\rho\right]. \tag{3.41}$$

The importance of the characteristic function comes from the Fourier-Weyl relation, which states that given a density operator ρ, we can write it as the phase space integral

$$\rho = \frac{1}{\pi}\int_{\mathbb{C}} \chi_\rho(\alpha)D(-\alpha)\mathrm{d}^2\alpha, \tag{3.42}$$

and hence, χ_ρ yields a complete knowledge of the quantum state ρ. The fact that $\mathrm{Tr}\,\rho = 1$ amounts to

$$\chi_\rho(0) = 1 \tag{3.43}$$

and the hermiticity of ρ manifests itself as

$$\overline{\chi_\rho(\alpha)} = \chi_\rho(-\overline{\alpha}). \tag{3.44}$$

Example (Characteristic function of the vacuum). The characteristic function of a single-mode vacuum state can be simply calculated as

$$\chi_{|0\rangle\langle 0|}(\alpha) = \mathrm{Tr}[D(\alpha)\,|0\rangle\langle 0|] = \langle 0|\,D(\alpha)\,|0\rangle = e^{-\frac{|\alpha|^2}{2}}. \tag{3.45}$$

For a multimode state ρ, the characteristic function is defined analogously with the multimode Weyl operator:

$$\chi_\rho(\boldsymbol{\alpha}) \coloneqq \mathrm{Tr}\left[D(\boldsymbol{\alpha})\rho\right]. \tag{3.46}$$

An important fact is that composing density operators from multiple subsystems amounts to a product in the level of characteristic functions, i.e., for density operators on ρ_A, ρ_B on supported on subsystems A, B, we can write their composite characteristic function as

$$\chi_{\rho_A\otimes\rho_B}(\boldsymbol{\alpha}_A \oplus \boldsymbol{\alpha}_B) = \chi_\rho(\boldsymbol{\alpha}_A)\,\chi_\sigma(\boldsymbol{\alpha}_B). \tag{3.47}$$

Example (Characteristic function of centralized pure Gaussian states). We know that a d-mode centralized pure Gaussian state can be written as

$$\rho = U\,|0\rangle\langle 0|\,U^\dagger, \tag{3.48}$$

where U is the unitary operator of a Gaussian gate. Denote the (complex form) symplectic matrix of this Gaussian gate by S. Then, the characteristic function can be calculated directly as

$$\chi_\rho(\boldsymbol{\alpha}) = \mathrm{Tr}\left[D(\boldsymbol{\alpha})G\,|0\rangle\langle 0|\,G^\dagger\right] = \mathrm{Tr}\left[\underbrace{G^\dagger D(\boldsymbol{\alpha})G}_{=D((S^{-1}\breve{\boldsymbol{\alpha}})_{1:d})}\,|0\rangle\langle 0|\right], \tag{3.49}$$

where the subscript $1:d$ denotes taking the first d elements of the vector. Therefore, using the characteristic function of the vacuum, we know that

$$\chi_\rho(\boldsymbol{\alpha}) = e^{-\frac{1}{4}\breve{\boldsymbol{\alpha}}^\dagger (S^{-1})^\dagger S^{-1}\breve{\boldsymbol{\alpha}}} \overset{\mathbb{1}=K^2}{=} e^{-\frac{1}{4}\breve{\boldsymbol{\alpha}}^\dagger (S^{-1})^\dagger K^2 S^{-1}\breve{\boldsymbol{\alpha}}} \overset{SKS^\dagger=K}{=} e^{-\frac{1}{4}\breve{\boldsymbol{\alpha}}^\dagger KSS^\dagger K\breve{\boldsymbol{\alpha}}} = e^{-\frac{1}{4}\breve{\boldsymbol{\alpha}}^\dagger K\sigma^{(\mathrm{c})}K\breve{\boldsymbol{\alpha}}}, \tag{3.50}$$

since the covariance matrix of the state $G\,|0\rangle\langle 0|\,G^\dagger$ can be written as

$$\sigma^{(\mathrm{c})} = SS^\dagger. \tag{3.51}$$

Similar statement holds for mixed Gaussian states, by replacing the covariance matrix. One important observation is that—as the characteristic function fully reproduces the density operator—Gaussian states only depend on their first and second statistical moments.

3.3.1 Wigner function

Definition 3.3.2. The Wigner function of a d-mode state ρ is defined as

$$W_\rho(\boldsymbol{\alpha}) = \frac{1}{\pi^{2d}}\int_{\mathbb{C}^d} e^{\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}-\overline{\boldsymbol{\alpha}}\boldsymbol{\beta}}\chi_\rho(\boldsymbol{\beta})\,\mathrm{d}^{2d}\boldsymbol{\beta}. \tag{3.52}$$

Remark. The inclusion of the term $e^{\boldsymbol{\alpha}\overline{\boldsymbol{\beta}}-\overline{\boldsymbol{\alpha}}\boldsymbol{\beta}}$ in the definition feels somewhat odd, but it is well-motivated, as this integral can be interpreted as a Fourier transformation defined on the phase space. Let us consider $d = 1$ for now, and write $\alpha = x + ip$ and $\beta = u + iv$, so we can write

$$\alpha\overline{\beta} - \overline{\alpha}\beta = 2i(pu - xv), \tag{3.53}$$

and hence,

$$W_\rho(x+ip) = \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{2i(pu-xv)}\chi_\rho(u+iv)\,\mathrm{d}u\mathrm{d}v = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ipu}e^{-ixv}\chi_\rho(u+iv)\,\mathrm{d}u\mathrm{d}v. \tag{3.54}$$

It can be shown, that—up to a factor $\frac{1}{2^{2d}}$—the Wigner function can be normalized, i.e.,

$$\frac{1}{2^{2d}}\int_{\mathbb{C}^d} W_\rho(\boldsymbol{\alpha})\,\mathrm{d}^{2d}\boldsymbol{\alpha} = 1, \tag{3.55}$$

but it is not necessarily non-negative, hence, it can not be scaled to a probability distribution over $\mathbb{C}$ in general (e.g., for non-vacuum Fock basis states). For this reason—when we normalize it—we call the Wigner function a *quasiprobability distribution*.

Example (Wigner function of a centralized pure Gaussian state). The Wigner function of a Gaussian state is just a Gaussian function, which can be shown by evaluating the Gaussian integral using Eq. (3.50):

$$W_\rho(\boldsymbol{\alpha}) = \frac{1}{\pi^{2d}} \int_{\mathbb{C}^d} e^{-\breve{\boldsymbol{\alpha}}^\dagger K \breve{\boldsymbol{\beta}}} e^{-\frac{1}{4}\breve{\boldsymbol{\beta}}^\dagger K \sigma^{(\mathrm{c})} K \breve{\boldsymbol{\beta}}} \, \mathrm{d}^{2d}\boldsymbol{\beta} = \frac{2^d}{\pi^d \sqrt{\det \sigma^{(\mathrm{c})}}} e^{-\breve{\boldsymbol{\alpha}}^\dagger (\sigma^{(\mathrm{c})})^{-1} \breve{\boldsymbol{\alpha}}}. \tag{3.56}$$

Note, that this is indeed normalized up to a factor of 2^{-2d}, as

$$\int_{\mathbb{C}^d} e^{-\breve{\boldsymbol{\alpha}}^\dagger (\sigma^{(\mathrm{c})})^{-1} \breve{\boldsymbol{\alpha}}} \mathrm{d}^{2d}\boldsymbol{\alpha} = (2\pi)^d \sqrt{\det \sigma^{(\mathrm{c})}}. \tag{3.57}$$

3.3.2 s-ordered characteristic functions

In quantum optics literature, one can find more versions of the characteristic function, which is related to the different orderings one could obtain from them.

Definition 3.3.3 (s-ordered characteristic function)**.** Given a density operator ρ, the *s-ordered characteristic function* is defined as

$$\chi_\rho^{(s)}(\boldsymbol{\alpha}) = \chi_\rho(\boldsymbol{\alpha}) e^{\frac{s}{2}\|\boldsymbol{\alpha}\|^2}. \tag{3.58}$$

We are interested in the cases $s = -1, 0, 1$, and when we talk about "the" characteristic function, we usually refer to the case $s = 0$. The exponential term is motivated by the following relations:

$$D(\boldsymbol{\alpha}) = e^{\boldsymbol{\alpha}\hat{a}^\dagger} e^{-\overline{\boldsymbol{\alpha}} a} e^{-\frac{1}{2}\|\boldsymbol{\alpha}\|^2} \qquad (s = +1), \tag{3.59}$$

$$D(\boldsymbol{\alpha}) = e^{-\overline{\boldsymbol{\alpha}} a} e^{\boldsymbol{\alpha}\hat{a}^\dagger} e^{+\frac{1}{2}\|\boldsymbol{\alpha}\|^2} \qquad (s = -1). \tag{3.60}$$

The first relation we have already seen in Eq. (3.5), and the second one can be acquired analogously. We are primarily interested in these because these are the generating function of normal and anti-normal ordered expectation values of ladder operators:

$$\begin{aligned} \mathrm{Tr}\left[(\hat{a}_1^\dagger)^{n_1} \dots (\hat{a}_d^\dagger)^{n_d} a_1^{m_1} \dots a_d^{m_d} \rho\right] &= (-1)^{\boldsymbol{m}} \left(\frac{\partial}{\partial \boldsymbol{\alpha}}\right)^{\boldsymbol{n}} \left(\frac{\partial}{\partial \overline{\boldsymbol{\alpha}}}\right)^{\boldsymbol{m}} \chi_\rho^{(1)}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=0}, \qquad \text{(normal ordering)} \\ \mathrm{Tr}\left[(\hat{a}_1^\dagger)^{m_1} \dots (\hat{a}_d^\dagger)^{m_d} a_1^{n_1} \dots a_d^{n_d} \rho\right] &= (-1)^{\boldsymbol{m}} \left(\frac{\partial}{\partial \boldsymbol{\alpha}}\right)^{\boldsymbol{n}} \left(\frac{\partial}{\partial \overline{\boldsymbol{\alpha}}}\right)^{\boldsymbol{m}} \chi_\rho^{(-1)}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=0}. \qquad \text{(anti-normal ordering)} \end{aligned} \tag{3.61}$$

Here, we used the multi-index notation. Similarly, in the $s = 0$ case, the same differential gives the normalized sum of expectation values of all possible orderings of n creation and m annihilation operator, the *symmetric ordering*. Hence, the cases $s = -1, 0, 1$ correspond to the expectation values of products of ladder operators in anti-normal, symmetric and normal ordering, respectively.

The following question arises: what are these expectation values of Gaussian states, i.e., what does Eq. (3.61) produce when we input the characteristic function of a Gaussian state? For this, we need to determine arbitrary differentials of Gaussian functions at $\boldsymbol{\alpha} \equiv 0$. For expressing these expectation values more compactly, we will use the following quantity, called the *hafnian*.

Definition 3.3.4 (Hafnian)**.** The *hafnian* of a $2m$-by-$2m$ symmetric matrix A is defined as

$$\mathrm{haf}(A) \coloneqq \sum_{p \in \mathrm{PM}([2m])} \prod_{(i,j)\in p} A_{ij}, \tag{3.62}$$

where $\mathrm{PM}([2m])$ denotes the perfect matchings of the set $[2m]$.

Proposition 9. Consider a symmetric matrix $\Sigma \in \mathbb{R}^{n\times n}$ and a nat-string $\boldsymbol{n} \in \mathbb{N}_0^n$. Then

$$\left(\frac{\partial}{\partial \boldsymbol{x}}\right)^{\boldsymbol{n}} \exp\left(\frac{1}{2}\boldsymbol{x}^T \Sigma \boldsymbol{x}\right)\Big|_{\boldsymbol{x}=0} = \mathrm{haf}(\mathrm{r}_{\boldsymbol{n}}(\Sigma)). \tag{3.63}$$

Proof. Consider first the case when $\boldsymbol{n} = (1, \dots, 1)$. Expand the exponential as

$$\exp\left(\frac{1}{2}\boldsymbol{x}^T \Sigma \boldsymbol{x}\right) = \sum_{n=0}^{\infty} \frac{1}{2^n n!} \left(\sum_{i,j=1}^{n} \Sigma_{ij} x_i x_j\right) \tag{3.64}$$

Now, we are searching for the coefficient of $x_1 \cdots x_n$, which is obviously 0 when n is odd, because the x_i terms appear in pairs in the formula. If $n = 2m$ is even, then we need to account for all the pairings of the indices $[2m] \coloneqq (1, \cdots, 2m)$. However, a given pairing can be obtained $2^n n!$ times canceling the denominator, and hence, the coefficient of $x_1 \cdots x_n$ is

$$\sum_{p \in \mathrm{PM}([2m])} \prod_{(i,j) \in p} \Sigma_{ij} = \mathrm{haf}(\Sigma). \tag{3.65}$$

To generalize this further, consider $\boldsymbol{n} \in \{0,1\}^n$, and let I denote the indices where $\boldsymbol{n}$ is 1. Then, we can decompose

$$\exp\left(\frac{1}{2}\boldsymbol{x}^T \Sigma \boldsymbol{x}\right) = \exp\left(\frac{1}{2}\sum_{\substack{i \in I \\ j \in I}} \Sigma_{ij} x_i x_j\right) \exp\left(\frac{1}{2}\left(\sum_{\substack{i \notin I \\ j \in I}} \Sigma_{ij} x_i x_j + \sum_{\substack{i \in I \\ j \notin I}} \Sigma_{ij} x_i x_j + \sum_{\substack{i \notin I \\ j \notin I}} \Sigma_{ij} x_i x_j\right)\right) \tag{3.66}$$

We know, that the second exponential does not contribute, as differentiation would bring down x_i terms with $i \notin I$, and hence these would not contribute. Hence, in this case we would get

$$\left(\frac{\partial}{\partial \boldsymbol{x}}\right)^{\boldsymbol{n}} \exp\left(\frac{1}{2}\boldsymbol{x}^T \Sigma \boldsymbol{x}\right)\Big|_{\boldsymbol{x}=0} = \mathrm{haf}(\mathrm{r}_{\boldsymbol{n}}(\Sigma)). \tag{3.67}$$

The multiplicities can be handled by, e.g., treating a variable x_i with n_i multiplicity as n_i independent variables $x_i^{(1)}, \cdots, x_i^{(n_i)}$. □

3.4 Glauber-Sudarshan P-representation, Husimi Q-function

Definition 3.4.1 (*s*-ordered quasiprobability distribution)**.** The s-ordered quasiprobability distribution is defined as

$$W_\rho^{(s)}(\boldsymbol{\alpha}) = \frac{1}{\pi^{2d}} \int_{\mathbb{C}^d} e^{\boldsymbol{\alpha}\overline{\boldsymbol{\beta}} - \overline{\boldsymbol{\alpha}}\boldsymbol{\beta}} \chi_\rho^{(s)}(\boldsymbol{\beta}) \mathrm{d}^{2d}\boldsymbol{\beta}. \tag{3.68}$$

In the $s = 0$ case, we recover the previously introduced Wigner function.

Theorem 3.4.2 (Glauber-Sudarshan P-representation of a quantum state)**.** Given a d-mode density operator ρ, we can write it as the following weighted integral over the phase space:

$$\rho = \int_{\mathbb{C}^d} P_\rho(\boldsymbol{\alpha}) \left|\boldsymbol{\alpha}\right\rangle\!\left\langle\boldsymbol{\alpha}\right| \mathrm{d}^{2d}\boldsymbol{\alpha}, \tag{3.69}$$

where $P_\rho(\boldsymbol{\alpha}) = W^{(+1)}(\boldsymbol{\alpha})$ is called the *P-function*.

Proof. We will not prove this theorem here, but the interested reader could find it in Ref. [38]. □

Analogously to the P-function, which is defined from the $s = +1$ characteristic function, the Q-function is defined using the $s = -1$ case:

Definition 3.4.3 (Husimi Q-function)**.** The *Q-function* of a d-mode state with density operator ρ is defined as

$$Q_\rho(\boldsymbol{\alpha}) = W_\rho^{(-1)}(\boldsymbol{\alpha}). \tag{3.70}$$

The Q-function can simply be written as

$$Q_\rho(\boldsymbol{\alpha}) = \frac{1}{\pi^d} \left\langle\boldsymbol{\alpha}\right| \rho \left|\boldsymbol{\alpha}\right\rangle. \tag{3.71}$$

As the characteristic function, the Q-function completely determines the quantum state.

Note, that P-functions are not necessarily functions in the usual sense. A standard example is the P-function of a single-mode Fock basis state:

Proposition 10 (P-function of single-mode Fock states)**.**

$$P_{|n\rangle\langle n|}(\alpha) = \frac{e^{|\alpha|^2}}{n!} \left(\frac{\partial^2}{\partial\alpha\partial\overline{\alpha}}\right)^n \delta^{(2)}(\alpha). \tag{3.72}$$

Proof. We can write

$$|n\rangle\langle n| = \int_{\mathbb{C}} P(\alpha)\,|\alpha\rangle\langle\alpha|\,\mathrm{d}^2\alpha. \tag{3.73}$$

We can take its overlap with the coherent state $|\beta\rangle\langle\beta|$ as

$$|\langle n|\beta\rangle|^2 = \frac{|\beta|^n}{n!}e^{-|\beta|^2} = \int_{\mathbb{C}} P(\alpha)|\langle\alpha|\beta\rangle|^2\mathrm{d}^2\alpha = \int_{\mathbb{C}} P(\alpha)e^{-|\alpha-\beta|^2}\mathrm{d}^2\alpha. \tag{3.74}$$

Substituting Eq. (3.72) we get

$$\begin{aligned}
\int_{\mathbb{C}} P(\alpha)e^{-|\alpha-\beta|^2}\mathrm{d}^2\alpha &= \frac{1}{n!}\int_{\mathbb{C}} e^{-|\alpha-\beta|^2}e^{|\alpha|^2}\left(\frac{\partial^2}{\partial\alpha\partial\overline{\alpha}}\right)^n \delta^{(2)}(\alpha)\mathrm{d}^2\alpha \\
&= \frac{e^{|\alpha|^2}}{n!}\int_{\mathbb{C}} \delta^{(2)}(\alpha)\left(\frac{\partial^2}{\partial\alpha\partial\overline{\alpha}}\right)^n e^{|\alpha|^2}e^{-|\alpha-\beta|^2}\mathrm{d}^2\alpha \\
&= \frac{1}{n!}\left(\frac{\partial^2}{\partial\alpha\partial\overline{\alpha}}\right)^n e^{|\alpha|^2}e^{-|\alpha-\beta|^2}|_{\alpha=0} \\
&= \frac{e^{|\beta|^2}}{n!}\left(\frac{\partial^2}{\partial\alpha\partial\overline{\alpha}}\right)^n e^{\alpha\overline{\beta}+\overline{\alpha}\beta}|_{\alpha=0} \\
&= \frac{e^{|\beta|^2}}{n!}|\beta|^n.
\end{aligned} \tag{3.75}$$

□

Proposition 11 (Q-function of centralized Gaussian states)**.** The Q-function of a d-mode centralized Gaussian state ρ with complex covariance matrix $\sigma^{(\mathrm{c})}$ is

$$Q_\rho(\boldsymbol{\alpha}) = \frac{1}{\pi^d\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}\breve{\boldsymbol{\alpha}}^\dagger\Sigma^{-1}\breve{\boldsymbol{\alpha}}\right), \tag{3.76}$$

where

$$\Sigma \coloneqq \frac{1}{2}(\sigma^{(\mathrm{c})} + \mathbb{1}). \tag{3.77}$$

Proof. The formula can be obtained by using the definition $Q_\rho(\boldsymbol{\alpha}) = W_\rho^{(-1)}(\boldsymbol{\alpha})$ and performing the Gaussian integral of the characteristic function (similarly as in Eq. (3.56)):

$$Q_\rho(\boldsymbol{\alpha}) = \frac{1}{\pi^{2d}}\int_{\mathbb{C}^d} e^{-\breve{\boldsymbol{\alpha}}^\dagger K\breve{\boldsymbol{\beta}}}e^{-\frac{1}{4}\breve{\boldsymbol{\beta}}^\dagger K\sigma^{(\mathrm{c})}K\breve{\boldsymbol{\beta}}}e^{-\frac{1}{4}\breve{\boldsymbol{\alpha}}^\dagger\breve{\boldsymbol{\alpha}}}\mathrm{d}^{2d}\boldsymbol{\beta} = \frac{1}{\pi^d\sqrt{\det\Sigma}}e^{\frac{1}{2}\breve{\boldsymbol{\beta}}^\dagger\Sigma^{-1}\breve{\boldsymbol{\beta}}}. \tag{3.78}$$

□

3.5 Exercises

Exercise 3.5.1. Prove Eq. (3.4)! *Hint: Write* $|\alpha\rangle = \sum_{n=0}^\infty c_n|n\rangle$ *and use Eq.* (3.2) *to obtain a relation for* c_n*!*

Solution. Let

$$|\alpha\rangle = \sum_{n=0}^\infty c_n|n\rangle, \qquad a|n\rangle = \sqrt{n}\,|n-1\rangle. \tag{3.79}$$

Using the eigenstate property $a|\alpha\rangle = \alpha|\alpha\rangle$ we get

$$a|\alpha\rangle = \sum_{n=0}^\infty c_n\sqrt{n}\,|n-1\rangle = \sum_{n=0}^\infty c_{n+1}\sqrt{n+1}\,|n\rangle \tag{3.80}$$

while

$$\alpha|\alpha\rangle = \sum_{n=0}^\infty \alpha\,c_n|n\rangle. \tag{3.81}$$

Equating coefficients of $|n\rangle$ yields the recurrence

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}}\, c_n \quad \Longrightarrow \quad c_n = \frac{\alpha^n}{\sqrt{n!}}\, c_0. \tag{3.82}$$

Normalize $|\alpha\rangle$:

$$1 = \langle\alpha|\alpha\rangle = \sum_{n=0}^{\infty} |c_n|^2 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \;\Rightarrow\; |c_0| = e^{-|\alpha|^2/2}. \tag{3.83}$$

Choosing the conventional real positive phase for c_0, $c_0 = e^{-|\alpha|^2/2}$. Hence,

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle\,. \tag{3.84}$$

Exercise 3.5.2. Plot the Wigner function of a single-mode vacuum state in Piquasso!

Solution. Use `https://app.piquasso.com/`!

Exercise 3.5.3. Plot the Wigner function of a single-mode coherent state in Piquasso!

Solution. Use `https://app.piquasso.com/`!

Exercise 3.5.4. Calculate the Q-function of a single-mode coherent state!

Solution. Using Eq. (3.71) and Eq. (3.12) we can write

$$Q_{|\alpha\rangle\langle\alpha|}(\beta) = \frac{1}{\pi}|\langle\beta|\alpha\rangle|^2 = \frac{1}{\pi} e^{-|\alpha-\beta|^2}. \tag{3.85}$$

Exercise 3.5.5. Calculate the P-function of a single-mode coherent state!

Solution. First, we need to calculate the characteristic function of $\rho = |\alpha\rangle\langle\alpha|$. This is

$$\chi_{|\alpha\rangle\langle\alpha|}(\beta) = \mathrm{Tr}[D(\beta)\,|\alpha\rangle\langle\alpha|] = \langle\alpha|\, D(\beta)\,|\alpha\rangle \overset{\text{Eq. (3.24)}}{=} e^{\overline{\alpha}\beta - \alpha\overline{\beta}} e^{-\frac{|\beta|^2}{2}}, \tag{3.86}$$

and hence,

$$\chi^{(+1)}_{|\alpha\rangle\langle\alpha|}(\beta) = \chi_{|\alpha\rangle\langle\alpha|}(\beta)^{-\frac{|\beta|^2}{2}} = e^{\overline{\alpha}\beta - \alpha\overline{\beta}}. \tag{3.87}$$

We know that

$$P_{|\alpha\rangle\langle\alpha|}(\gamma) = \frac{1}{\pi^2}\int_{\mathbb{C}} e^{\gamma\overline{\beta} - \overline{\gamma}\beta} e^{\overline{\alpha}\beta - \alpha\overline{\beta}} \mathrm{d}^2\beta = \frac{1}{\pi^2}\int_{\mathbb{C}} e^{\overline{(\alpha-\gamma)}\beta - (\alpha-\gamma)\overline{\beta}} \mathrm{d}^2\beta. \tag{3.88}$$

Using the techniques from the proof of Proposition 7, we can write that

$$\int_{\mathbb{C}} e^{\overline{(\alpha-\gamma)}\beta - (\alpha-\gamma)\overline{\beta}} \mathrm{d}^2\beta = \pi^2 \delta^{(2)}(\alpha - \gamma), \tag{3.89}$$

and therefore

$$P_{|\alpha\rangle\langle\alpha|}(\gamma) = \delta^{(2)}(\alpha - \gamma). \tag{3.90}$$

Note, that this formula could also have been easily obtained from Theorem 3.4.2 by guessing.

Exercise 3.5.6. Calculate the Q-function of a single-mode squeezed state!

Solution. Use Proposition 11 and the covariance matrix from Exercise 2.7.6! We can write

$$\Sigma = \frac{1}{2}(\sigma^{(\mathrm{c})} + \mathbb{1}) = \frac{1}{2}\begin{bmatrix} \cosh(2r) + 1 & -e^{i\phi}\sinh(2r) \\ -e^{-i\phi}\sinh(2r) & \cosh(2r) + 1 \end{bmatrix}, \tag{3.91}$$

where $z = re^{i\phi}$. By direct calculation, we also know that

$$\det\Sigma = (\cosh(2r) + 1)^2 - \sinh^2(2r) = \cosh^2(r), \tag{3.92}$$

and the inverse Σ^{-1} is

$$\Sigma^{-1} = \begin{bmatrix} 1 & e^{i\phi}\tanh(r) \\ e^{-i\phi}\tanh(r) & 1 \end{bmatrix}. \tag{3.93}$$

Chapter 4

Gaussian Boson Sampling

4.1 Particle number detection probabilities of Gaussian states

The concept of Gaussian Boson Sampling (GBS), i.e., the photon detection measurement of a general Gaussian state, was first introduced in Refs. [30, 42]. Similar to conventional Boson Sampling, the classical resources required to take a sample from a complex probability distribution of Gaussian photonic states tend to scale exponentially with the number of photons involved, making the problem classically intractable. The complexity of taking a sample from a non-displaced Gaussian state is characterized by the evaluation of a matrix function called the *hafnian* [43], which we will show in this section.

Consider a d-mode Gaussian state with a density operator ρ. The probability of detecting the particle numbers $\boldsymbol{n} = (n_1, \ldots, n_d)$ is given by Born's rule:

$$p(\boldsymbol{n}) \coloneqq \mathrm{Tr}\left[\rho \left|\boldsymbol{n}\right\rangle\!\left\langle\boldsymbol{n}\right|\right]. \tag{4.1}$$

In this analysis, we will use the phase space formulation of quantum optics introduced in Chapter 3. According to Theorem 3.4.2, we can write

$$\left|\boldsymbol{n}\right\rangle\!\left\langle\boldsymbol{n}\right| = \int_{\mathbb{C}^d} P_{\left|\boldsymbol{n}\right\rangle\!\left\langle\boldsymbol{n}\right|}(\boldsymbol{\alpha}) \left|\alpha\right\rangle\!\left\langle\alpha\right| \mathrm{d}\boldsymbol{\alpha}, \tag{4.2}$$

where $\mathrm{d}\boldsymbol{\alpha} = \prod_{i=1}^{d} \mathrm{d}^2\alpha_i$. Hence, the probability can be rewritten as

$$p(\boldsymbol{n}) = \pi^d \int_{\mathbb{C}^d} Q_\rho(\boldsymbol{\alpha}) P_{\left|\boldsymbol{n}\right\rangle\!\left\langle\boldsymbol{n}\right|}(\boldsymbol{\alpha}) \mathrm{d}\boldsymbol{\alpha}. \tag{4.3}$$

Adapting Eq. (3.72) for multiple modes, we get that

$$P_{\left|\boldsymbol{n}\right\rangle\!\left\langle\boldsymbol{n}\right|}(\boldsymbol{\alpha}) = \frac{e^{\|\boldsymbol{\alpha}\|^2}}{\boldsymbol{n}!} \left(\frac{\partial^2}{\partial\boldsymbol{\alpha}\partial\overline{\boldsymbol{\alpha}}}\right)^{\boldsymbol{n}} \delta^{(2d)}(\boldsymbol{\alpha}), \tag{4.4}$$

where $\delta^{(2d)}(\boldsymbol{\alpha}) \coloneqq \delta^{(2)}(\alpha_1) \cdots \delta^{(2)}(\alpha_d)$. Using Proposition 11, we can write

$$p(\boldsymbol{n}) = \frac{1}{\boldsymbol{n}!\sqrt{\det\Sigma}} \left(\frac{\partial^2}{\partial\boldsymbol{\alpha}\partial\overline{\boldsymbol{\alpha}}}\right)^{\boldsymbol{n}} e^{\|\boldsymbol{\alpha}\|^2} \exp\left(-\frac{1}{2}\breve{\boldsymbol{\alpha}}^\dagger \Sigma^{-1} \breve{\boldsymbol{\alpha}}\right)\Big|_{\boldsymbol{\alpha}=\boldsymbol{0}} = \frac{1}{\boldsymbol{n}!\sqrt{\det\Sigma}} \left(\frac{\partial^2}{\partial\boldsymbol{\alpha}\partial\overline{\boldsymbol{\alpha}}}\right)^{\boldsymbol{n}} \exp\left(\frac{1}{2}\breve{\boldsymbol{\alpha}}^T A \breve{\boldsymbol{\alpha}}\right)\Big|_{\boldsymbol{\alpha}=\boldsymbol{0}}. \tag{4.5}$$

where we denote

$$A \coloneqq X(\mathbb{1} - \Sigma^{-1}), \tag{4.6}$$

$$X \coloneqq \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}. \tag{4.7}$$

Here, X plays the role of flipping $\breve{\boldsymbol{\alpha}}^\dagger$ to $\breve{\boldsymbol{\alpha}}^T$, i.e., $\breve{\boldsymbol{\alpha}}^\dagger X = \breve{\boldsymbol{\alpha}}^T$ Finally, according to Proposition 9, we know that

$$\left(\frac{\partial^2}{\partial\boldsymbol{\alpha}\partial\overline{\boldsymbol{\alpha}}}\right)^{\boldsymbol{n}} \exp\left(\frac{1}{2}\breve{\boldsymbol{\alpha}}^T A \breve{\boldsymbol{\alpha}}\right)\Big|_{\boldsymbol{\alpha}=\boldsymbol{0}} = \mathrm{haf}(\mathrm{br}_{\boldsymbol{n}}\, A), \tag{4.8}$$

where $\mathrm{br}_{\boldsymbol{n}} \coloneqq \mathrm{r}_{\boldsymbol{n}\oplus\boldsymbol{n}}$. Putting it all together, we get

$$p(\boldsymbol{n}) = \frac{\mathrm{haf}(\mathrm{br}_{\boldsymbol{n}}\, A)}{\boldsymbol{n}!\sqrt{\det \Sigma}}. \tag{4.9}$$

When the Gaussian state is pure, A can be written as

$$A = \begin{bmatrix} B & \\ & \overline{B} \end{bmatrix}, \tag{4.10}$$

where

$$B = U \tanh(\boldsymbol{r}) U^T \tag{4.11}$$

for some d-dimensional unitary U corresponding to an interferometer and $\boldsymbol{r} \geq 0$ corresponding to a layer of squeezing gates. In this case—due the fact that the hafnian decomposes into products under direct summation—we can write $\mathrm{haf}(\mathrm{br}_{\boldsymbol{n}}\, A) = |\mathrm{haf}(\mathrm{r}_{\boldsymbol{n}}\, B)|^2$, yielding

$$p(\boldsymbol{n}) = \frac{|\mathrm{haf}(\mathrm{r}_{\boldsymbol{n}}\, B)|^2}{\boldsymbol{n}!\sqrt{\det \Sigma}}. \tag{4.12}$$

4.2 Properties of the hafnian and connection to graph theory

To connect to hafnian to the number of perfect matching in a graph, we need to recap some graph theory.

4.2.1 Recap: Graph theory

For completeness, we provide the definition of a graph here. For simplicity, we will not consider weighted/signed/directed graphs in this course.

Definition 4.2.1 (Graph)**.** A *graph* is an ordered pair $G = (V, E)$, where V is a finite set of *vertices* (or nodes), and $E \subseteq \{\{u, v\} \mid u, v \in V, u \neq v\}$ is a set of *edges* connecting pairs of distinct vertices.

Graphs can be represented via their adjacency matrix:

Definition 4.2.2 (Adjacency matrix)**.** A graph can be represented by its adjacency matrix A, an $|V|$-dimensional matrix with entries $A_{ij} = 1$ if $(i, j) \in E$, i.e., if there is an edge between the i-th and j-th vertices, otherwise $A_{ij} = 0$.

The adjacency matrix of an (undirected) graph be symmetric, i.e., $A^T = A$. The following definition encapsulates the notion of "all-to-all connected" graphs:

Definition 4.2.3 (Complete graph)**.** A graph is called *complete* if for any pair of vertices there is an edge connecting them.

Definition 4.2.4 (Subgraph density)**.** Given a graph $G = (V, E)$, and a subgraph $S = (V_S, E_S)$. The *density* of the subgraph S is the ratio defined as

$$d(S) = \frac{|E_S|}{\binom{|V_S|}{2}}.$$

Definition 4.2.5 (Bipartite graph)**.** A *bipartite graph* is such a graph whose vertices can be divided into two disjoint and independent sets, such that every edge connects a vertex in one set to one in the other set.

The adjacency matrix of such graph can be written as

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}. \tag{4.13}$$

The matrix B is sometimes called the *biadjacency matrix*.

Matchings are fundamental structures in graph theory, capturing the idea of vertex pairings. Among them, perfect matchings play a particularly important role, as they correspond to a complete pairing of the vertex set.

Definition 4.2.6 (Perfect matching)**.** Let $G = (V, E)$ be an undirected graph. A *perfect matching* $M \subseteq E$ is a set of edges such that every vertex $v \in V$ is incident to exactly one edge in M. Equivalently, M is a matching that covers all vertices of G. The set of perfect matchings is denoted by $\mathrm{PM}(G)$.

4.2.2 Perfect matchings and the hafnian

The permanent of a matrix comes into picture when we try to count the perfect matchings in a bipartite graph. For a bipartite graph, the number of perfect matchings is given by the permanent of its $n \times n$ bipartite adjacency matrix A. Each nonzero product in the definition of the permanent corresponds to a perfect matching, and the permanent counts all such matchings, i.e., given a bipartite graph G with adjacency matrix of the form

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}, \tag{4.14}$$

we can write that

$$\mathrm{per}(B) = |\mathrm{PM}(G)|. \tag{4.15}$$

Similarly, a key observation is that the hafnian of A equals the number of perfect matchings in the graph, i.e., given a bipartite graph G with adjacency matrix A, we can write that

$$\mathrm{haf}(A) = |\mathrm{PM}(G)|. \tag{4.16}$$

This is logical, as in the definition of the hafnian, we iterate over all perfect matching of the set $\mathrm{PM}([n])$ where n is the number of vertices, and a term in the sum only gives a 1 contribution the perfect matching from $\mathrm{PM}([n])$ is also a perfect matching of the graph G.

4.3 Finding dense subgraphs using Gaussian Boson Sampling

By embedding the adjacency matrix A into a GBS setup, one obtains a sampler that preferentially produces outcomes corresponding to subgraphs with a large number of perfect matchings. This property can be utilized in applications of GBS to graph problems, such as finding dense subgraphs, since graphs with many perfect matchings are typically dense. Consequently, GBS can accelerate heuristic classical algorithms that rely on subgraph sampling as a subroutine, including a stochastic algorithm for the densest k-subgraph problem [31], which is the following:

Problem (Densest k-subgraph problem)**.** Given a graph G and $k < |G|$, find a subgraph of k vertices with the largest density!

It has a natural connection to clustering problems with the goal of finding highly correlated subsets of data, having a wide range of applications. However, this problem is NP-hard [44], and approximation is also believed to be inefficient [45]. However, stochastic algorithms are believed to be more useful for finding dense k-subgraphs.

We will introduce a hybrid stochastic algorithm, which uses GBS to provide guesses for dense subgraphs, originally described in Ref. [31]. By previous discussions, the probability of observing a given photon configuration is proportional to the number of perfect matchings of the corresponding subgraph. Also, intuitively, a graph with many perfect matchings is expected to contain many edges. This has been made rigorous in Ref. [46].

For now, suppose that we can embed the adjacency matrix in the GBS circuit and we can obtain samples corresponding to subgraphs with a high number of perfect matchings, which we call exploration. However, we can also exploit the obtained samples, i.e., we can provide an enhanced strategy that tweaks the samples obtained from GBS to possibly achieve any improvement. Such tweaking strategy is given by Algorithm 1. This algorithm can be used within an optimization algorithm searching for dense subgraphs, e.g., in a simulated annealing algorithm. It is important to mention that from the resulting samples, we have to post-select the samples that correspond to k-subgraphs, i.e., samples that only contain 0s and 1s (called *collision-free* samples), and the number of 1s is exactly k. In such a sample, the indices where the 1s appear correspond to the indices of the vertices in the sampled subgraph.

Algorithm 1: GBS-Tweak

Input: A graph G with adjacency matrix A, and a subgraph S with adjacency matrix $\mathrm{r}_S A$, $d \coloneqq |G|$, $k \coloneqq |S|$, and l is the minimum number of vertices to keep.

Output: A k-vertex subgraph of G.

1. Generate R as an l-vertex subgraph of S with adjacency matrix $\mathrm{r}_R \mathrm{r}_S A$ according to the GBS distribution

$$p(R|S) \propto \left|\mathrm{haf}(\mathrm{r}_R \mathrm{r}_S A)\right|^2.$$

   Extend R by picking a uniform random number $m \in \{0, 1, \ldots, k-l-1\}$ of the remaining vertices from S, along with the corresponding edges. This defines the subgraph G_{keep} on $l+m$ vertices.

2. Generate T as a $(k-l)$-vertex subgraph of A with adjacency matrix $\mathrm{r}_T A$ according to the GBS distribution

$$p(T) \propto \left|\mathrm{haf}(\mathrm{r}_T A)\right|^2.$$

   Reduce T by randomly rejecting m of its vertices (and corresponding edges). This defines the subgraph G_{replace} on $k-l-m$ vertices. If G_{keep} and G_{replace} share any vertices, repeat this step.

3. Output the k-vertex subgraph

$$G_{\mathrm{keep}} \cup G_{\mathrm{replace}}.$$

For solving graph problems using GBS, we will need to embed graph adjacency matrices into the photonic quantum circuit. This can be done using the following decomposition:

Theorem 4.3.1 (Takagi–Autonne decomposition)**.** Let $A \in \mathbb{C}^{n\times n}$ be a complex symmetric matrix, i.e. $A^T = A$. Then there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and a diagonal matrix $D = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with nonnegative real entries such that

$$A = UDU^T.$$

The diagonal entries σ_i are uniquely determined up to ordering, and are the *singular values* of A.

Proof. We omit the proof of this theorem, but it can be found in Ref. [47]. □

With this, we can decompose any adjacency matrix to have the form as B from Eq. (4.11). Here, D corresponds to the layer of squeezing gates, and U corresponds to the interferometer matrix of the interferometer in the circuit. Notice, that the adjacency matrix can be scaled freely, which amounts to adjusting the average number of photons through the squeezing parameters.

4.4 Exercises

Exercise 4.4.1. Show that when the Gaussian state is pure, A can be written as

$$A = \begin{bmatrix} B & \\ & \overline{B} \end{bmatrix}. \tag{4.17}$$

Solution. We know that the covariance matrix of any pure d-mode Gaussian state can be written as

$$\sigma^{(\mathrm{c})} = SS^\dagger, \tag{4.18}$$

for $S \in \mathrm{Sp}^{(\mathrm{c})}(2d, \mathbb{R})$. By the Bloch-Messiah decomposition, we can write this as

$$\sigma^{(\mathrm{c})} = V \sigma^{(\mathrm{c})}_{\mathrm{squeezed}} V^\dagger, \tag{4.19}$$

where $\sigma^{(\mathrm{c})}_{\mathrm{squeezed}}$ corresponds to d independent squeezed states as initial states and has the covariance matrix

$$\sigma^{(\mathrm{c})}_{\mathrm{squeezed}} = \begin{bmatrix} \cosh \boldsymbol{r} & -\sinh \boldsymbol{r} \\ -\sinh \boldsymbol{r} & \cosh \boldsymbol{r} \end{bmatrix}. \tag{4.20}$$

Moreover, V has the form

$$V = \begin{bmatrix} U & \\ & \overline{U} \end{bmatrix}, \tag{4.21}$$

where $U \in \mathrm{U}(d)$ is an interferometer matrix. We can write that

$$\Sigma = \frac{1}{2}(\mathbb{1} + \sigma^{(\mathrm{c})}) = V\frac{1}{2}(\mathbb{1} + \sigma^{(\mathrm{c})}_{\mathrm{squeezed}})V^\dagger =: V\Sigma_{\mathrm{squeezed}}V^\dagger, \tag{4.22}$$

where $\Sigma_{\mathrm{squeezed}}$ is the Q-matrix of initial squeezed states, and hence

$$A = X(\mathbb{1} - \Sigma^{-1}) = XV(\mathbb{1} - \Sigma^{-1}_{\mathrm{squeezed}})V^\dagger. \tag{4.23}$$

According to Eq. (3.93), we know that the inverse of the Q-matrix of a single-mode squeezed state is

$$\begin{bmatrix} 1 & e^{i\phi}\tanh(r) \\ e^{-i\phi}\tanh(r) & 1 \end{bmatrix}, \tag{4.24}$$

and—for convenience—we can choose $\phi = \pi/2$ uniformly for all squeezing gates, which we can do freely, as this only amounts to a phase, and can be moved into the interferometer. With this adjustment, we can write

$$\Sigma^{-1}_{\mathrm{squeezed}} = \begin{bmatrix} \mathbb{1} & -\tanh(r) \\ -\tanh(r) & \mathbb{1} \end{bmatrix}, \tag{4.25}$$

and hence

$$A = XV \begin{bmatrix} & \tanh(r) \\ \tanh(r) & \end{bmatrix} V^\dagger = X \begin{bmatrix} \overline{U}\tanh(r)U^\dagger & \\ & U\tanh(r)U^T \end{bmatrix} \tag{4.26}$$

$$= \begin{bmatrix} \overline{U}\tanh(r)U^\dagger & \\ & U\tanh(r)U^T \end{bmatrix} =: \begin{bmatrix} B & \\ & \overline{B} \end{bmatrix}. \tag{4.27}$$

Exercise 4.4.2. Consider the graph with adjacency matrix

```
adjacency_matrix = np.array(
    [
        [0, 1, 0, 0, 0, 0, 0, 1, 1, 0],
        [1, 0, 1, 1, 1, 0, 1, 1, 0, 1],
        [0, 1, 0, 1, 0, 1, 0, 1, 1, 1],
        [0, 1, 1, 0, 0, 0, 0, 1, 1, 0],
        [0, 1, 0, 0, 0, 0, 0, 1, 1, 1],
        [0, 0, 1, 0, 0, 0, 1, 0, 0, 0],
        [0, 1, 0, 0, 0, 1, 0, 0, 0, 1],
        [1, 1, 1, 1, 1, 0, 0, 0, 0, 1],
        [1, 0, 1, 1, 1, 0, 0, 0, 0, 1],
        [0, 1, 1, 0, 1, 0, 1, 1, 1, 0],
    ]
)

```

and embed it into a Piquasso circuit using the `Graph` instruction! Sample subgraphs using `ParticleNumberMeasurement`! Convert the samples with `Result.to_subgraph_nodes` to vertex indices, and filter out the collision-free samples corresponding to k-vertex subgraphs! To maximize the number of useful samples, adjust the `mean_photon_number` parameter of `Graph`!

Solution. The following code should do the trick:

```
import numpy as np
import networkx as nx
import piquasso as pq


adjacency_matrix = np.array(
    [
        [0, 1, 0, 0, 0, 0, 0, 1, 1, 0],
        [1, 0, 1, 1, 1, 0, 1, 1, 0, 1],
        [0, 1, 0, 1, 0, 1, 0, 1, 1, 1],
        [0, 1, 1, 0, 0, 0, 0, 1, 1, 0],
        [0, 1, 0, 0, 0, 0, 0, 1, 1, 1],
        [0, 0, 1, 0, 0, 0, 1, 0, 0, 0],
        [0, 1, 0, 0, 0, 1, 0, 0, 0, 1],
        [1, 1, 1, 1, 1, 0, 0, 0, 0, 1],
        [1, 0, 1, 1, 1, 0, 0, 0, 0, 1],
```

```
        [0, 1, 1, 0, 1, 0, 1, 1, 1, 0],
    ]
)


def postselect(subgraph_nodes, k):
    result = []

    for subgraph in subgraph_nodes:
        if len(subgraph) == len(np.unique(subgraph)) and len(subgraph) == k:
            result.append(subgraph)

    return result


shots = 1000
k = 6

with pq.Program() as program:
    pq.Q() | pq.Vacuum() | pq.Graph(
        adjacency_matrix, mean_photon_number=k / len(adjacency_matrix)
    ) | pq.ParticleNumberMeasurement()


simulator = pq.GaussianSimulator(d=len(adjacency_matrix))

result = simulator.execute(program, shots=shots)

dense_subgraph_nodes = result.to_subgraph_nodes()

filtered_dense_subgraph_nodes = postselect(dense_subgraph_nodes, k)

graph = nx.from_numpy_array(adjacency_matrix)

for i, x in enumerate(filtered_dense_subgraph_nodes):
    subgraph = graph.subgraph(x)
    print(f"Graph {i}: {subgraph} (density: {nx.density(subgraph)})")

```

Exercise 4.4.3. Implement the GBS-Tweak algorithm 1 using `networkx` and the previous exercise!

Solution. See attached notebook!

Exercise 4.4.4. Use the GBS-Tweak algorithm to implement a simple simulated annealing algorithm for finding dense k-subgraphs!

Solution. See attached notebook!

Appendix A

Recap: Basics of quantum mechanics

As a reference, this chapter revisits the minimal basics of quantum mechanics needed for this thesis, which is done by presenting several postulates. For a more detailed discussion of quantum mechanics and quantum computing, the reader may to visit Ref. [2].

A.1 Hilbert spaces

In quantum mechanics, it is often necessary to work in a vector space with complex numbers as scalars in potentially infinite dimensions. Such a construction requires a generalization of the standard Euclidean vector space which can be done by introducing vector spaces with complex numbers as the base field. These vector spaces are also naturally endowed with an inner product, the direct analog of the dot product of Euclidean spaces, which allows us to define angles and distances. This leads us to the concept of the Hilbert space, an essential tool for describing quantum mechanical phenomena.

Definition A.1.1 (Complex inner product space)**.** A complex inner product space V is a complex vector space with an inner product $\langle \cdot | \cdot \rangle$, which satisfies the following properties:

- Conjugation symmetry:
$$\langle x|y\rangle = \overline{\langle y|x\rangle}, \qquad x, y \in V. \tag{A.1}$$
- Linearity:
$$\langle x|ay + bz\rangle = a\,\langle x|y\rangle + b\,\langle x|z\rangle\,, \qquad x, y, z \in V,\ a, b \in \mathbb{C}, \tag{A.2}$$
- Positive definiteness:
$$\begin{aligned} &\langle x|x\rangle \geq 0, \quad \forall\, x \in V, \\ &\langle x|x\rangle = 0 \Leftrightarrow x = 0. \end{aligned} \tag{A.3}$$

In quantum mechanics, complex inner product spaces are assumed to be Cauchy-complete, meaning that Cauchy sequences will converge inside the vector space. We call Cauchy-complete complex inner product spaces *complex Hilbert spaces*:

Definition A.1.2 (Complex Hilbert space)**.** A complex Hilbert space $\mathcal{H}$ is a complex inner product space that is complete with respect to the distance function induced by the inner product, $d(x, y) = \sqrt{\langle x-y|x-y\rangle}$. Completeness means that every Cauchy sequence in $\mathcal{H}$ has a limit that is also in $\mathcal{H}$.

Using the norm induced from the inner product and denoted by $||\cdot||$, convergence, and thus infinite sums can be defined in Hilbert spaces (as opposed to arbitrary infinite-dimensional vector spaces).

Definition A.1.3. Let $\mathcal{H}$ be a Hilbert space and I a set of indices. The set $\{e_i \in \mathcal{H}\}_{i\in I}$ is called a *basis* if

1. $\{e_i\}_{i\in I}$ is a linearly independent set,
2. $\forall\, v \in \mathcal{H} \setminus \{0\} : \exists\, J \subseteq I$, where J is at most countably infinite in cardinality and a corresponding unique $\{c_j : c_j \in \mathbb{C}\}_{j\in J}$ set such that:
$$v = \sum_{j\in J} c_j e_j, \tag{A.4}$$

Moreover, if $\langle e_i|e_j\rangle = \delta_{ij}$ for all $i, j \in I$, then it is called an *orthonormal basis* (ONB).

Then number of elements in a basis is an essential property of vector spaces:

Definition A.1.4 (Dimension)**.** The dimension of a Hilbert space is equal to the cardinality of its basis.

Finite-dimensional complex Hilbert spaces are isomorphic to $\mathbb{C}^d$ for some $d \geq 1$. Therefore, the following example is often used when dealing with quantum systems using finite-dimensional Hilbert spaces:

Example. The vector space $\mathbb{C}^d$ endowed with the dot product as the inner product

$$\begin{aligned} \langle \cdot | \cdot \rangle : \mathbb{C}^d \times \mathbb{C}^d &\to \mathbb{C}, \\ (v, w) &\mapsto \sum_{j=1}^{d} \overline{v}_j w_j \end{aligned} \tag{A.5}$$

is a Hilbert space.

The space of square-integrable functions endowed with the usual inner product is an important infinite-dimensional example of a Hilbert space.

Example (Square-integrable functions). Let us define

$$\langle f|g\rangle \coloneqq \int_{-\infty}^{\infty} \overline{f(x)} g(x) \mathrm{d}x, \tag{A.6}$$

where $f, g : \mathbb{R} \to \mathbb{C}$. The set of square-integrable functions $L^2(\mathbb{R})$ forms a Hilbert space with the inner product defined in Eq. (A.6).

While some Hilbert spaces considered in this thesis are finite-dimensional, the underlying Hilbert space of the bosonic system corresponding to a photonic quantum computer has a countably infinite orthonormal basis. It should be noted, that in quantum mechanics, all Hilbert spaces are assumed to be separable, i.e., to admit an at most countably infinite orthonormal basis. Therefore, in this thesis, all Hilbert spaces are assumed to be separable.

In quantum mechanics, vectors are often denoted using angle brackets as $|\psi\rangle \in \mathcal{H}$, called *ket* vectors. To introduce their counterpart, consider the set of linear operators $\mathcal{H} \to \mathbb{C}$, i.e., *linear forms*. By Riesz's representation theorem, for every linear form on a Hilbert space $f_\phi : \mathcal{H} \to \mathbb{C}$, there exists a vector $|\phi\rangle \in \mathcal{H}$ such that for all $|\psi\rangle \in \mathcal{H}$, we have $f_\phi(|\psi\rangle) = \langle\phi|\psi\rangle$. For brevity, we denote the linear form $f_\phi : \mathcal{H} \to \mathbb{C}$ by $\langle\phi|$, which is called the *bra* vector corresponding to the ket vector $|\phi\rangle$. This notation is very convenient, since the expressions $\langle\phi|, |\psi\rangle, \langle\phi|\psi\rangle$ automatically encode the correspondence between linear forms and vectors in a Hilbert space. This convention is called the *bra-ket notation* or *Dirac notation*.

A.2 Quantum states

In quantum mechanics, it is convenient to define the set of states over a separable Hilbert space:

Definition A.2.1 (State space)**.** The state space on a separable Hilbert space $\mathcal{H}$ is

$$\mathcal{S}(\mathcal{H}) \coloneqq \big\{\rho \in \mathrm{B}(\mathcal{H}) : \operatorname{Tr}\rho = 1, \rho \geq 0\big\}, \tag{A.7}$$

where $\mathrm{B}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$, and $\rho \geq 0$ means that ρ is self-adjoint linear operator with a non-negative spectrum. Elements of $\mathcal{S}(\mathcal{H})$ are called *density operators*.

The eigenvalues of a density operator $\rho \in \mathcal{S}(\mathcal{H})$ form a probability distribution with $\dim \mathcal{H}$ elements. Quantum states correspond to density operators over a separable Hilbert space, which is the gist of the first postulate of quantum mechanics:

Postulate 1. Associated to any quantum system there corresponds a separable Hilbert space known as the *state space* of the quantum system, and *quantum states* are completely described by density operators over this Hilbert space.

Definition A.2.2 (Pure state)**.** A density operator describes a *pure state* if it is a rank one projection. Equivalently, a density operator ρ describes a pure state if and only if $\rho = \rho^2$.

If a quantum state is pure, then its density operator can be written in the form $\rho = |\psi\rangle\langle\psi|$. Therefore, pure quantum states correspond to unit vectors $|\psi\rangle$ on a Hilbert space $\mathcal{H}$ with equivalence relation $|\psi\rangle \sim |\psi'\rangle$ for any $|\psi'\rangle = e^{i\alpha}|\psi\rangle$, where $\alpha \in [0, 2\pi)$.

A.3 Evolution

For describing the time evolution of a closed quantum system, the unitary group over the Hilbert space is essential:

Definition A.3.1 (Unitary group over a Hilbert space)**.** The surjective bounded linear operators preserving the inner product in a Hilbert space $\mathcal{H}$ are called unitary operators, i.e., U preserves the inner product if $\langle U\psi, U\phi\rangle = \langle \psi, \phi\rangle$ for every $|\psi\rangle, |\phi\rangle \in \mathcal{H}$. The set of unitary operators forms a group, denoted by $\mathrm{U}(\mathcal{H})$, and is called the *unitary group* over $\mathcal{H}$.

With this definition, one can write down the second postulate of quantum mechanics, which prescribes how closed quantum systems evolve:

Postulate 2. Time evolution of a closed quantum system modeled by the Hilbert space $\mathcal{H}$ can be described by unitary operators over $\mathcal{H}$.

It is important to note, that over a finite-dimensional Hilbert space, any unitary operator can be written in the form

$$\hat{U} = \exp\left(i\hat{H}\right), \tag{A.8}$$

where $\hat{H}$ is a self-adjoint operator over $\mathcal{H}$ called the Hamilton operator.

A.4 Measurement

So far we have postulated how to model closed quantum systems and how to describe their evolution. However, we also need to extract information from these systems via measurements, which makes them interact with their environment and thus no longer described by a unitary evolution.

Modeling measurements on a quantum system can be done by positive operator-valued measures (POVMs):

Definition A.4.1 (POVM for countable outcomes)**.** Measurements on a quantum system modeled by a finite-dimensional Hilbert-space $\mathcal{H}$ can be described by positive operator-valued measures (POVMs), given by

$$\mathrm{POVM}(\mathcal{X}, \mathcal{H}) \coloneqq \left\{ \{M_x\}_{x\in\mathcal{X}} \in \mathrm{Lin}(\mathcal{H}) : 0 \leq M_x \leq \mathbb{1}_{\mathcal{H}},\ \sum_{x\in\mathcal{X}} M_x = \mathbb{1}_{\mathcal{H}} \right\}, \tag{A.9}$$

where $\mathcal{X}$ is a countably infinite set corresponding to the possible measurement outcomes.

Note that we have described the definition of the POVMs applicable to countable outcomes, but one can easily generalize it to uncountable sets if required. We note, that when the contituent operators are all projections, the POVM is called a projection-valued measure (PVM). Importantly, the measurement probabilities are given by Born's rule, which is the third postulate:

Postulate 3 (Born's rule)**.** Measurements are described by POVMs, and indexed by the possible outcomes of the measurements. When the system is in the state $\rho \in \mathcal{S}(\mathcal{H})$ and a measurement described by a POVM $M = \{M_x\}_{x\in\mathcal{X}} \in \mathrm{POVM}(\mathcal{H})$ is performed on the system then the probability of obtaining the outcome x is given by

$$\mathrm{Prob}_{\rho,M}(\mathrm{outcome} = x) = \mathrm{Tr}\,\rho M_x. \tag{A.10}$$

The measurement is a way to extract information from a quantum system. While the density operator encodes all our knowledge about the system, the measurements can be considered questions to the density operator. One may easily verify that Born's rule with the definition of POVM specifies a probability distribution in measurement outcomes, since

$$\begin{aligned} \mathrm{Prob}_{\rho,M}(\mathrm{outcome} = x) &= \mathrm{Tr}\,\rho M_x \geq 0, \\ \sum_{x\in\mathcal{X}} \mathrm{Prob}_{\rho,M}(\mathrm{outcome} = x) &= \mathrm{Tr}\,\rho \sum_{x\in\mathcal{X}} M_x = 1. \end{aligned} \tag{A.11}$$

A.5 Composite systems

The concept of the tensor product enables us to study systems with multiple particles, hence, it is essential in constructing the vector space for modeling photonic quantum states:

Definition A.5.1 (Tensor product of Hilbert spaces)**.** Consider Hilbert space $\mathcal{H}$ and $\mathcal{K}$. Let us denote

$$\mathcal{H} \otimes \mathcal{K} \coloneqq F(\mathcal{H} \times \mathcal{K})/R, \tag{A.12}$$

where $F(\mathcal{H} \times \mathcal{K})$ denotes the free vector space generated by $\mathcal{H} \times \mathcal{K}$, R is the subspace spanned by

$$R = \operatorname{span}_{\mathbb{C}} \left(\left\{ \begin{array}{r} (v_1 + v_2, w) - (v_1, w) - (v_2, w), \\ (v, w_1 + w_2) - (v, w_1) - (v, w_2), \\ (sv, w) - s(v, w), \\ (v, sw) - s(v, w) \end{array} \middle| \begin{array}{r} \forall\, v_1, v_2, v \in \mathcal{H}, \\ w_1, w_2, w \in \mathcal{K}, \\ s \in \mathbb{C} \end{array} \right\} \right), \tag{A.13}$$

and the quotient $F(\mathcal{H} \times \mathcal{K})/R$ is defined by the equivalence relation $\sim_R$ as

$$a \sim_R b \iff a - b \in R. \tag{A.14}$$

In the following, we introduce the mapping $\otimes : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}$, which takes elements of $\mathcal{H}$ and $\mathcal{K}$ to their *tensor product* in $\mathcal{H} \otimes \mathcal{K}$:

$$v \otimes w \coloneqq (v, w) \in \mathcal{H} \otimes \mathcal{K}. \tag{A.15}$$

The universal property may also give an equivalent definition of the tensor product:

Definition A.5.2 (Tensor product via universal property)**.** Let $\mathcal{H}$ and $\mathcal{K}$ be two finite-dimensional Hilbert spaces. The tensor product of $\mathcal{H}$ and $\mathcal{K}$ denoted by $\mathcal{H} \otimes \mathcal{K}$ is a vector space with a bilinear map

$$\otimes : \mathcal{H} \times \mathcal{K} \to \mathcal{H} \otimes \mathcal{K}, \tag{A.16}$$

which has the universal property, i.e., if $f : \mathcal{H} \times \mathcal{K} \to \mathcal{H}'$, then there exists a linear map (up to an isomorphism) $\hat{f} : \mathcal{H} \otimes \mathcal{K} \to \mathcal{H}'$ such that $\otimes \circ \hat{f} := f$, where $\mathcal{H}'$ is a finite-dimensional Hilbert space.

While this definition does not guarantee existence, for the tensor product for Hilbert spaces, Definition A.5.1 gives a construction. The uniqueness may be easily verified from the universal property by diagrammatic arguments.

The concept of the tensor product enables us to write down the final, fourth postulate of quantum mechanics:

Postulate 4. The Hilbert space of a composite quantum system is modeled by the tensor product of the Hilbert spaces of the constituent quantum systems.

A basis of the tensor product Hilbert spaces can be written as tensor products of all basis vectors from the constituent Hilbert spaces:

Proposition 12. Let $\mathcal{H}$ and $\mathcal{K}$ be two finite n and m dimensional, Hilbert spaces, with basis vectors $\{e_i\}_{i=1}^{n} \subseteq \mathcal{H}$ and $\{f_j\}_{j=1}^{m} \subseteq \mathcal{K}$. The tensor product of $\mathcal{H}$ and $\mathcal{K}$, denoted by $\mathcal{H} \otimes \mathcal{K}$ is the $n \cdot m$ dimensional Hilbert space with the basis vectors:

$$\{e_i \otimes f_j\}_{i,j=1}^{n,m}, \tag{A.17}$$

with the inner product given on the basis vectors as:

$$\langle e_i \otimes f_j | e_k \otimes f_l \rangle_{\mathcal{H}\otimes\mathcal{K}} = \langle e_i | e_k \rangle_{\mathcal{H}} \langle f_j | f_l \rangle_{\mathcal{K}}. \tag{A.18}$$

Note that the tensor product of multiple Hilbert spaces can be defined inductively from the tensor product of 2 Hilbert spaces, since the tensor product of Hilbert spaces is also a Hilbert space. Furthermore, one can easily verify the associativity of the tensor product. Moreover, for the tensor product of Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n$ the following notation is used:

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n = \bigotimes_{i=1}^{n} \mathcal{H}_i, \tag{A.19}$$

For the special case when

$$\mathcal{H}_1 = \mathcal{H}_2 = \cdots = \mathcal{H}_n = \mathcal{H}, \tag{A.20}$$

a shorter notation can be used:

$$\mathcal{H}^{\otimes n} \coloneqq \bigotimes_{i=1}^{n} \mathcal{H}. \tag{A.21}$$

We can now also define the tensor product of linear operators:

Definition A.5.3. Let $\mathcal{H}$ and $\mathcal{K}$ be two finite, n and m dimensional Hilbert spaces with orthonormal basis vectors $\{e_i\}_{i=1}^{n} \subseteq \mathcal{H}$ and $\{f_j\}_{j=1}^{m} \subseteq \mathcal{K}$. Let $X : \mathcal{H} \to \mathcal{H}'$ and $Y : \mathcal{K} \to \mathcal{K}'$ be two linear operators, where $\mathcal{H}'$ and $\mathcal{K}'$ are also finite-dimensional Hilbert spaces. Then $X \otimes Y$ is a linear operator on $\mathcal{H} \otimes \mathcal{K}$ that acts on the elements of the ONB in $\mathcal{H} \otimes \mathcal{K}$ in the following way:

$$(X \otimes Y)(e_i \otimes f_j) = (Xe_i) \otimes (Yf_j).$$

Consider now a composite system that can be modeled by $\mathcal{H} = \bigotimes_{i=1}^{n} \mathcal{H}_i$, where $\mathcal{H}_i$ correspond to the constituent subsystems. If there is no interaction between the subsystems, then a time evolution on the composite system can be described as

$$\hat{U} = \bigotimes_{i=1}^{n} \hat{U}_i = \bigotimes_{i=1}^{n} \exp\left(-i\hat{H}_i\right), \tag{A.22}$$

where $\hat{U}_i \in \mathrm{U}(\mathcal{H}_i)$ are the time evolution operators corresponding to the subsystems with Hamilton operators $\hat{H}_i$. With this definition, the global Hamilton operator of the system can be written as

$$\hat{H} = \sum_{i}^{n} \hat{H}_i \otimes \bigotimes_{j,i\neq j}^{n} \mathbb{1}_j, \tag{A.23}$$

where $\mathbb{1}_j \in \mathrm{Lin}(\mathcal{H}_j)$ denotes the identity on $\mathrm{Lin}(\mathcal{H}_j)$.

A.6 Quantum mechanics of the harmonic oscillator

As a reference, a brief description of the simple quantum harmonic oscillator starting from the Schrödinger equation is presented in this section. The Schrödinger equation is a partial differential equation governing the wave function of a closed quantum system, in our case the quantum harmonic oscillator. The time evolution is governed by the Hamilton operator

$$\hat{H} \coloneqq \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\right). \tag{A.24}$$

To get the stationary wave functions corresponding to this Hamiltonian, we need to solve the time-independent Schrödinger equation of the quantum harmonic oscillator, which is

$$\hat{H}\psi(x) \coloneqq \frac{1}{2}\left(-\frac{\partial^2}{\partial x^2} + x^2\right)\psi(x) = E_\psi \psi(x), \tag{A.25}$$

where $E_\psi \in \mathbb{R}$ is called the eigenvalue corresponding to the eigenfunction (or eigenstate) $\psi : \mathbb{R} \to \mathbb{C}$. By defining the position and momentum operators

$$\hat{x} \coloneqq x\cdot, \tag{A.26}$$

$$\hat{p} \coloneqq -i\frac{\partial}{\partial x}, \tag{A.27}$$

we can rewrite the Hamilton-operator $\hat{H}$ as

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2). \tag{A.28}$$

One can factor this operator into linear parts as

$$\hat{H} = \frac{\hat{x} - i\hat{p}}{\sqrt{2}}\frac{\hat{x} + i\hat{p}}{\sqrt{2}} - \frac{i}{2}[\hat{x}, \hat{p}] = \frac{\hat{x} - i\hat{p}}{\sqrt{2}}\frac{\hat{x} + i\hat{p}}{\sqrt{2}} + \frac{1}{2}\mathbb{1}, \tag{A.29}$$

where the identity $[\hat{x}, \hat{p}] = i\mathbb{1}$ can be used, which can be easily verified by direct calculation. Let us define the *ladder operators* as

$$a \coloneqq \frac{\hat{x} + i\hat{p}}{\sqrt{2}}, \tag{A.30}$$

$$\hat{a}^\dagger \coloneqq \frac{\hat{x} - i\hat{p}}{\sqrt{2}}, \tag{A.31}$$

where a is called the *annihilation operator*, and $\hat{a}^\dagger$ is called the *creation operator*. With this notation, we can rewrite the Hamiltonian in the following, simpler form:

$$\hat{H} = \hat{a}^\dagger a + \frac{1}{2}\mathbb{1}. \tag{A.32}$$

Proposition 13. Any function $\psi_0 : \mathbb{R} \to \mathbb{C}$ for which

$$a\psi_0 \equiv 0 \tag{A.33}$$

is an eigenfunction of $\hat{H}$ with eigenvalue $E_{\psi_0} = \frac{1}{2}$.

Proof. By direct calculation, we get

$$\hat{H}\psi_0(x) = \hat{a}^\dagger \underbrace{a\psi_0(x)}_{=0} + \frac{1}{2}\psi_0(x) = \frac{1}{2}\psi_0(x). \tag{A.34}$$

□

Physically, ψ_0 is the eigenstate corresponding to the vacuum, and the annihilation operator applied to it yields 0.

Proposition 14. The functions $\psi_n \coloneqq \left(\hat{a}^\dagger\right)^n \psi_0$ also serve as solutions of the time-independent Schrödinger equation corresponding to the quantum harmonic oscillator, with eigenvalues

$$E_{\psi_n} \coloneqq n + \frac{1}{2}. \tag{A.35}$$

Proof. By direct calculation, we get that

$$[a, \left(\hat{a}^\dagger\right)^n] = n\left(\hat{a}^\dagger\right)^{n-1}, \tag{A.36}$$

and hence we get that

$$[\hat{H}, \left(\hat{a}^\dagger\right)^n] = n\left(\hat{a}^\dagger\right)^n. \tag{A.37}$$

Substituting into the time-independent Schrödinger equation we get

$$\hat{H}\left(\hat{a}^\dagger\right)^n \psi_0(x) = [\hat{H}, \left(\hat{a}^\dagger\right)^n]\psi_0(x) + \left(\hat{a}^\dagger\right)^n \underbrace{\hat{H}\psi_0(x)}_{=\frac{1}{2}\psi_0(x)} = \left(n + \frac{1}{2}\right)\left(\hat{a}^\dagger\right)^n \psi_0(x). \tag{A.38}$$

□

Therefore, solutions of the time-independent Schrödinger equation can be easily generated from ψ_0. By solving the differential equation $a\psi_0(x) = 0$, we get that

$$\psi_0(x) = A_0 e^{-\frac{x^2}{2}}, \qquad (A_0 \in \mathbb{C}). \tag{A.39}$$

To fulfill the normalization condition, we can set $A_0 = \pi^{-1/4}$. Moreover, we can directly calculate that

$$\left(\hat{a}^\dagger\right)^n e^{-\frac{x^2}{2}} = 2^{-n/2}\left(\hat{x} - \frac{\partial}{\partial x}\right)^n e^{-\frac{x^2}{2}} \tag{A.40}$$

generate the solutions corresponding to higher eigenvalues. Therefore, we know that

$$\psi_n(x) = A_n H_n(x) e^{-\frac{x^2}{2}}, \tag{A.41}$$

where $A_n \in \mathbb{R}$ is some normalization factor, $H_n(x)$ is some polynomial in x, and we already know that $A_0 = \pi^{-1/4}$ and $H_0(x) = 1$. By inserting this into the time-independent Schrödinger equation, we get the following differential equation:

$$\frac{\mathrm{d}^2 H_n(x)}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}H_n(x)}{\mathrm{d}x} + 2nH_n(x) = 0, \tag{A.42}$$

which is exactly the Hermite differential equation [48]. The solutions of these differential equations are the Hermite polynomials defined by

$$H_n(x) = (-1)^n e^{x^2}\frac{\mathrm{d}^n}{\mathrm{d}x^n}e^{-x^2}. \tag{A.43}$$

The Hermite polynomials also fulfill the following orthogonality relations

$$\int_{\mathbb{R}} H_n(x)H_m(x)e^{-x^2}\mathrm{d}x = \sqrt{\pi}2^n n!\delta_{nm}, \tag{A.44}$$

and hence

$$\langle\psi_n|\psi_m\rangle = |A_n|^2\sqrt{\pi}2^n n!\delta_{nm}, \tag{A.45}$$

meaning that $\{\psi_n\}_{n=0}^{\infty}$ forms an orthogonal basis in $L^2(\mathbb{R})$ so one can normalize ψ_n by $A_n = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}$. Hence, the normalized wave functions are

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}}H_n(x)e^{-\frac{x^2}{2}}. \tag{A.46}$$

Appendix B

First quantization

In this appendix, we discuss a more elementary formalism compared to the one used in Chapter 1, which is called *first quantization*. In the previous discussions, we used *second quantization* (also called occupation number representation).

In many physical situations, one considers a system composed of several *indistinguishable particles* that are either bosons or fermions. The indistinguishability manifests itself in the symmetry properties of the composite state. For *bosonic* systems, all the possible states describing n particles are totally symmetric. More precisely, let $\mathcal{H}$ be the Hilbert space of a single particle, let $\pi \in S_n$, where S_n is the permutation group of n elements, and consider the unitary action $U_\pi : \mathcal{H}^{\otimes n} \to \mathcal{H}^{\otimes n}$ (where we denote $\mathcal{H}^{\otimes n} \coloneqq \mathcal{H} \otimes \cdots \otimes \mathcal{H}$) defined by

$$U_\pi \psi = U_\pi \, \phi_1 \otimes \cdots \otimes \phi_n = \phi_{\pi(1)} \otimes \cdots \otimes \phi_{\pi(n)}. \tag{B.1}$$

Generally, a vector $\psi \in \mathcal{H}^{\otimes n}$ is *totally symmetric*, if the following is true:

$$\forall\, \pi \in S_n \;:\; U_\pi \psi = \psi. \tag{B.2}$$

For bosons, the n-particle Hilbert space is the subspace of $\mathcal{H}^{\otimes n}$ defined by

$$\mathcal{H}^{\otimes n}|_{\text{symm}} \coloneqq \big\{ \psi \in \mathcal{H}^{\otimes n} : U_\pi \psi = \psi,\ \forall\, \pi \in S_n \big\}. \tag{B.3}$$

It is useful to introduce the notation $\phi_1 \vee \cdots \vee \phi_n$ for the *symmetrized tensor product*:

$$\phi_1 \vee \cdots \vee \phi_n \coloneqq \frac{1}{\sqrt{n!}} \sum_{\pi \in S_n} \phi_{\pi(1)} \otimes \cdots \otimes \phi_{\pi(n)}, \tag{B.4}$$

which yields totally symmetric vectors, i.e., $U_\pi \phi_1 \vee \cdots \vee \phi_n = \phi_1 \vee \cdots \vee \phi_n$ for any $\pi \in S_n$ and $\phi_i \in \mathcal{H}$, and hence $\phi_1 \vee \cdots \vee \phi_n \in \mathcal{H}^{\otimes n}|_{\text{symm}}$. It can also be shown that vectors of this form generate the space of totally symmetric vectors, i.e.,

$$\mathcal{H}^{\otimes n}|_{\text{symm}} = \mathcal{H}^{\vee n} \coloneqq \text{span}\big\{ \phi_1 \vee \cdots \vee \phi_n \in \mathcal{H}^{\otimes n}|_{\text{symm}} \big\}. \tag{B.5}$$

Thus, $\mathcal{H}^{\otimes n}|_{\text{symm}}$ is called the *n-particle Hilbert space* of bosonic systems.

Using Eq. (B.4), it can be easily verified that

$$\langle \phi_1 \vee \cdots \vee \phi_n | \psi_1 \vee \cdots \vee \psi_n \rangle = \text{per}\left([\langle \phi_i | \psi_j \rangle]_{i,j=1}^n \right), \tag{B.6}$$

where per denotes the permanent from Eq. (1.52), and $[\langle \phi_i | \psi_j \rangle]_{i,j=1}^n$ is the matrix constructed from the inner products $\langle \phi_i | \psi_j \rangle$. Using this formula, one can construct an ONB in $\mathcal{H}^{\vee n}$:

Proposition 15 (Fock basis)**.** Suppose that $B = \{e_1, \ldots, e_d\}$ is an orthonormal basis in $\mathcal{H}$. Then

$$B_n \coloneqq \left\{ \frac{1}{\sqrt{n_1! \cdots n_d!}} e_{i_1} \vee \cdots \vee e_{i_n} : i_1 \le \cdots \le i_n \right\} \tag{B.7}$$

forms an orthonormal basis in $\mathcal{H}^{\vee n}$, where $n_1, \ldots, n_d$ denote the repetitions of the indices in increasing order.

Remark. For any unit vector $\psi \in \mathcal{H}$, one can consider the symmetrized tensor product $\psi^{\vee n} \in \mathcal{H}^{\vee n}$,

$$\psi^{\vee n} = \frac{n!}{\sqrt{n!}} \psi^{\otimes n} = \sqrt{n!}\, \psi^{\otimes n}, \tag{B.8}$$

and for its norm on $\mathcal{H}^{\otimes n}$ (induced from the inner product on $\mathcal{H}$), one gets

$$\|\psi^{\vee n}\|^2 = \langle \psi^{\vee n} | \psi^{\vee n} \rangle = n!\,. \tag{B.9}$$

In this way, a unit vector can be constructed by $\frac{1}{\sqrt{n!}} \psi^{\vee n}$, since

$$\frac{1}{n!} \|\psi^{\vee n}\|^2 = \frac{1}{n!} n! = 1. \tag{B.10}$$

Consider now a bosonic quantum system of $\dim \mathcal{H}$ independent optical modes. The Hilbert space consisting of all the n-particle Hilbert spaces is called the *bosonic Fock space*, i.e., it can be written as the direct sum of the symmetric n-particle Hilbert spaces [49]:

$$\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\vee n}. \tag{B.11}$$

The closure is needed to ensure that the bosonic Fock space is also a Hilbert space. We usually refer to $\mathcal{H}$ as the *one-particle Hilbert space*, whereas, as mentioned, $\mathcal{H}^{\vee n}$ is the n-particle Hilbert space. Here, by convention, $\mathcal{H}^{\vee 0} \cong \mathbb{C}$, and it corresponds to the vacuum.

The formalism of second quantization provides an intuitive language for describing bosonic systems and is also used in the description of photonic quantum computation. In this formalism, the role of the bosonic annihilation and creation operators is essential.

Definition B.0.1 (Bosonic creation and annihilation operator)**.** Let $\mathcal{H}$ be a one-particle Hilbert space of a bosonic system. The creation operator corresponding to a one-particle vector $\psi \in \mathcal{H}$ is the unbounded operator $\hat{a}^\dagger(\psi)$ on $\mathcal{F}(\mathcal{H})$ that maps an n-particle vector onto a $(n+1)$-particle vector in the following manner:

$$\hat{a}^\dagger(\psi) : \phi_1 \vee \cdots \vee \phi_n \mapsto \psi \vee \phi_1 \vee \cdots \vee \phi_n\,. \tag{B.12}$$

The annihilation operator $a(\psi)$ is an unbounded operator on $\mathcal{F}(\mathcal{H})$ that maps an n-particle vector onto an $(n-1)$-particle vector in the following manner:

$$a(\psi) : \phi_1 \vee \cdots \vee \phi_n \mapsto \sum_{i=1}^{n} \langle \psi | \phi_i \rangle\, \phi_1 \vee \cdots \vee \phi_{i-1} \vee \phi_{i+1} \vee \cdots \vee \phi_n\,. \tag{B.13}$$

Creation and annihilation operators are often called the *ladder operators*.

Let us consider a simple case when the one-particle Hilbert space $\mathcal{H}_1$ is one-dimensional, which corresponds to a system with a single bosonic mode. With the choice of a unit vector ψ, the action of $\hat{a}^\dagger(\psi)$ can be represented as

$$\hat{a}^\dagger(\psi) : \psi^{\vee n} \mapsto \psi^{\vee (n+1)}, \tag{B.14}$$

and for the annihilation operator as

$$a(\psi) : \psi^{\vee n} \mapsto n \psi^{\vee (n-1)}, \tag{B.15}$$

where $\psi^{\vee 0} \coloneqq \psi^{\otimes 0} \coloneqq 1$ and using $\psi^{\vee n} = \sqrt{n!}\, \psi^{\otimes n}$, we get that

$$\begin{aligned} \hat{a}^\dagger(\psi) : \sqrt{n!}\, \psi^{\otimes n} &\mapsto \sqrt{(n+1)!}\, \psi^{\otimes (n+1)}, \\ a(\psi) : \sqrt{n!}\, \psi^{\otimes n} &\mapsto \sqrt{n}\sqrt{n!}\, \psi^{\otimes (n-1)}\,. \end{aligned} \tag{B.16}$$

One may also use the occupation number notation:

$$\begin{aligned} |n\rangle &\coloneqq \frac{1}{\sqrt{n!}}\, \psi^{\vee n}, \\ \hat{a}^\dagger &\coloneqq \hat{a}^\dagger(\psi), \\ a &\coloneqq a(\psi), \end{aligned} \tag{B.17}$$

rendering Eq. (B.14) simpler by

$$\begin{aligned} \hat{a}^\dagger |n\rangle &= \sqrt{n+1}\, |n+1\rangle\,, \\ a\, |n\rangle &= \sqrt{n}\, |n-1\rangle\,. \end{aligned} \tag{B.18}$$

Note that $|n\rangle$ is normalized by construction, i.e., $\langle n|n\rangle = 1$. From the definition, it is also obvious that $\langle n|m\rangle = \delta_{nm}$ for $n, m \in \mathbb{N}$, where δ_{nm} denotes the Kroenecker delta. The set of vectors

$$\{|n\rangle \in \mathcal{F}(\mathcal{H}_1) : n \in \mathbb{N}_0\} \tag{B.19}$$

constitutes an orthonormal basis in $\mathcal{F}(\mathcal{H}_1)$, where $\mathcal{H}_1$ is a one-dimensional Hilbert space $\mathbb{C}$. Generalizing the previous discussion, let us now consider a system with d bosonic modes. In this case, the bosonic Fock space is $\mathcal{F}(\mathcal{H})$, where $\mathcal{H}$ is a d-dimensional Hilbert space. Considering an ONB of the one-particle Hilbert space $\{e_1, \ldots, e_d\} \subset \mathcal{H}$, the d basis vectors can be assigned to the d bosonic modes in the system. Thus, similarly to the single-mode case, the Fock basis states can be defined by the basis in $\mathcal{F}(\mathcal{H})$ constructed from the ONB on the one-particle Hilbert space using Proposition 15:

$$|n_1, \ldots, n_d\rangle \coloneqq \frac{1}{\sqrt{n_1! \ldots n_d!}} e_{i_1} \vee \cdots \vee e_{i_n}. \tag{B.20}$$

For a fixed ONB in $\mathcal{F}(\mathcal{H})$, one may use the compact notation $\hat{a}_i^\dagger \coloneqq \hat{a}^\dagger(e_i)$, $a_i \coloneqq a(e_i)$. One can easily verify, that these operators act on the constructed ONB as

$$\begin{aligned} \hat{a}_j^\dagger\, |n_1, \ldots, n_j, \ldots, n_d\rangle &= \sqrt{n_j+1}\, |n_1, \ldots, n_j+1, \ldots, n_d\rangle\,, \\ \hat{a}_j\, |n_1, \ldots, n_j, \ldots, n_d\rangle &= \sqrt{n_j}\, |n_1, \ldots, n_j-1, \ldots, n_d\rangle\,. \end{aligned} \tag{B.21}$$

For the creation and annihilation operators, the following relations hold:

$$\begin{aligned} [a_i, \hat{a}_j^\dagger] &= \delta_{ij} \mathbb{1}, \\ [a_i, \hat{a}_j] &= [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0, \end{aligned} \tag{B.22}$$

for all $i, j \in [d]$.

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