

# **THE ROLE OF SYMMETRY**

## **IN QUANTUM ALGORITHMS FOR MANY-BODY SYSTEMS**

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**IVANA MIHÁLIKOVÁ, JOSEPH CARLSON, DUFF NEILL, IONEL STETCU**

# THE CHALLENGE: EXPONENTIAL HILBERT SPACE

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**N** spin-1/2 particles → Hilbert space dimension  $2^N$

**$2^{12}$**

12 spins  
4 096 states

**$2^{30}$**

30 spins  
1 073 741 824 states

**$2^{50}$**

50 spins  
1 125 899 906 842 624 states

Classical computers hit a wall. Quantum computers can help, and they help much more when they are guided to the relevant part of this space.

**KEY QUESTION: HOW DO WE FOCUS THE ALGORITHM ON THE PART OF THE HILBERT SPACE THAT ACTUALLY MATTERS?**

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**KEY QUESTION: HOW DO WE FOCUS THE ALGORITHM ON THE PART OF THE HILBERT SPACE THAT ACTUALLY MATTERS?**

Symmetries reduce the relevant Hilbert space, increase the spectral gap, and give quantum algorithms a cleaner problem to solve, leading to faster and more reliable convergence.

# WHY ARE SYMMETRIES IMPORTANT?

①

## SMALLER SPACE

Hilbert space decomposes into **independent subspaces** labelled by quantum numbers (total spin). Only one subspace is relevant, we can ignore the rest completely.

②

## LARGER GAP

Symmetry constraints increase the spectral gap  $\Delta$ , **the energy difference between the ground state and the first excited state.** A larger gap means faster convergence for quantum algorithms.

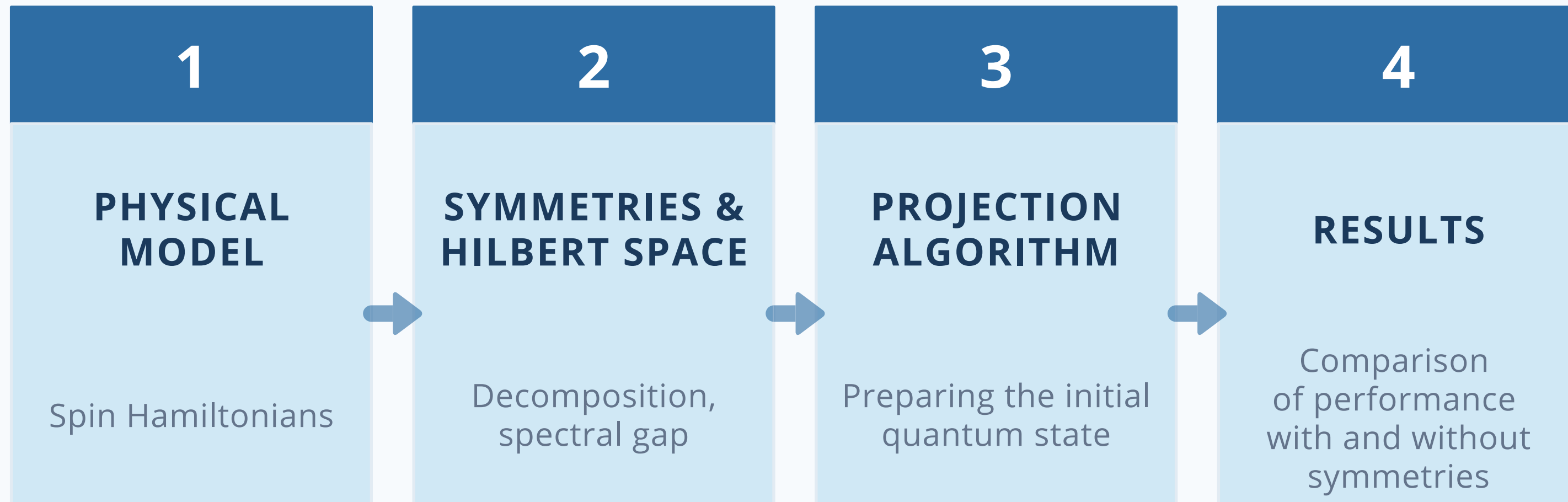
③

## BETTER FOCUS

Symmetries give algorithms a cleaner, more structured problem to solve, leading to **faster and more reliable convergence.**

# OVERVIEW

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# 1

## PHYSICAL MODELS

Two spin Hamiltonians with different symmetry structures

# THE HEISENBERG MODEL

## Heisenberg Spin-1/2

$$H = \mathcal{J} \sum_{\langle i,j \rangle} \sigma_i \cdot \sigma_j = \mathcal{J} \sum_{\langle i,j \rangle} \left( \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z \right) \text{ (sum over nearest-neighbour pairs)}$$

**4×3** square lattice with periodic boundary conditions — **12 spins** total

Full Hilbert space:  $2^{12} = \mathbf{4096}$  dimensions

### Symmetries:

- total spin  $J$ ,
- z-projection  $J_z$ ,
- translational symmetry,
- mirror symmetry (x and y)

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**The ground state is characterized by quantum numbers  $J = 0$  and  $J_z = 0$ .**

# THE NEUTRINO-INSPIRED MODEL

## All-to-All Random Couplings

$$H = \sum_{i < j} (1 - \cos \theta_{ij}) \sigma_i \cdot \sigma_j \quad (\text{all-to-all connections})$$

**12 spins** on a **4×3 lattice** with all-to-all random interactions

Full Hilbert space:  $2^{12} = \mathbf{4096}$  dimensions

**Coupling coefficient (1 - cos  $\theta_{ij}$ ):**  $\cos \theta_{ij}$  is randomly chosen from the range [-1, 1]

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**12 spins** on a **4×3 lattice** with all-to-all random interactions

Full Hilbert space:  $2^{12} = 4096$  dimensions

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# WHY THESE TWO MODELS?

## Heisenberg

- $J, J_z$
- Translational symmetry
- Mirror symmetry (x and y)
- Nearest-neighbor coupling
- Rich symmetry structure

VS

## Neutrino model

- $J, J_z$  only
- No spatial symmetries
- All-to-all coupling
- Minimal symmetry structure

The contrast lets us isolate the contribution of each symmetry and see exactly what it adds to algorithm performance.

# 2

## **SYMMETRIES AND HILBERT SPACE**

Decomposition, spectral gap, and fidelity

# SYMMETRY AS HILBERT SPACE DECOMPOSITION

When  $H$  commutes with a symmetry operator  $\rightarrow$  Hilbert space decomposes into independent subspaces, each labelled by a quantum number (e.g. total spin  $J$ ).

- States in different subspaces never mix under time evolution.
- If the ground state has  $J = 0$ , we restrict the entire algorithm to that subspace and ignore everything else.

**KEY INSIGHT: WE DO NOT NEED TO EXPLORE THE FULL HILBERT SPACE,  
SYMMETRY TELLS US EXACTLY WHERE TO LOOK.**

# SYMMETRY AS HILBERT SPACE DECOMPOSITION

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Symmetry gives us: smaller space, larger gap and more focused algorithm.

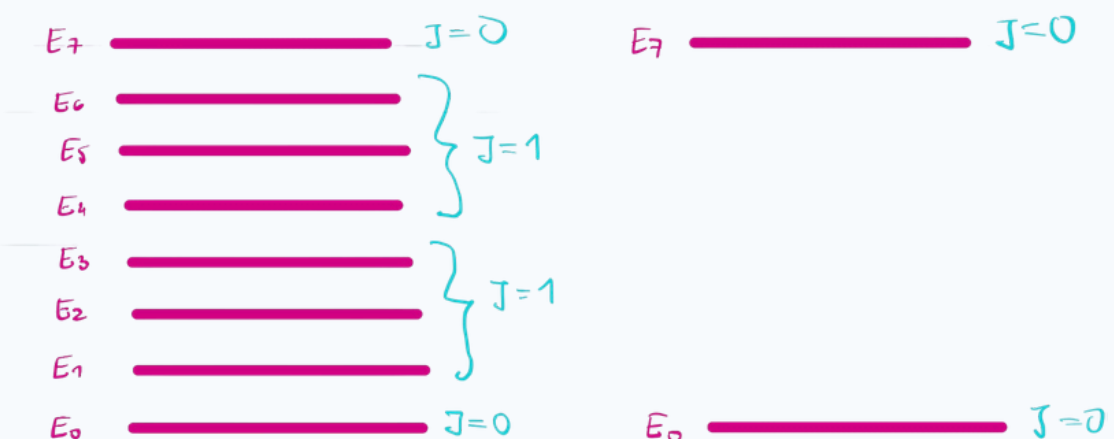
# SYMMETRY AS HILBERT SPACE DECOMPOSITION



## HEISENBERG MODEL

Spectral gap  $\Delta$  (energy difference between the ground state and the first excited state in the relevant symmetry sector):

Full space:  $\Delta \approx 7$   $\rightarrow$   $J = 0$  sector:  $\Delta \approx 28$



Symmetry gives us: smaller space, larger gap and more focused algorithm.

# MEASURING QUALITY: FIDELITY

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$$\text{Fidelity} = |\langle \psi | \varphi \rangle|^2$$

$\psi$  = prepared state

$\varphi$  = exact ground state

- Fidelity = **1** → **perfect match** with the exact ground state
- Fidelity = **0** → **states are completely orthogonal** (no overlap at all)
- Goal: push fidelity as close to 1 as possible at each step of the pipeline

# 3

## **INITIAL STATES AND PROJECTION ALGORITHM**

Preparing the initial quantum state

# STARTING POINT — PRODUCT STATES

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- Simple product states: tensor products of single-qubit states, easy to prepare with single-qubit gates only
- **Heisenberg model:** symmetrized Néel state; antiferromagnetic checkerboard pattern  $\uparrow \downarrow \uparrow \downarrow \dots$

$$|\psi_0\rangle = \frac{1}{\sqrt{2}} (|101001011010\rangle + |010110100101\rangle)$$

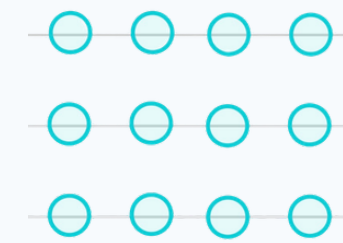


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$$|\psi_i\rangle = \bigotimes_{i=1}^N \exp\left(i\pi \mathbf{k}_i \cdot \frac{\boldsymbol{\sigma}_i}{2}\right) |0\rangle_i,$$



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- **Neutrino model:** flavor-momentum locked state; each spin aligned to its momentum vector (semiclassical ground state for large N)
- These states capture some physics but do not respect quantum symmetries

MODEL	INITIAL FIDELITY
Neutrino model	2.4%
Heisenberg model	8.5%



Low initial fidelity, but a good warm start for the projection step.

# THE PROJECTION ALGORITHM

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Our Hamiltonians commute with two operators: **the total spin  $J^2$  and its z-projection  $J_z$** . This means both  $J$  and  $J_z$  are conserved quantities. The ground state lives in the subspace where both  $J = 0$  and  $J_z = 0$ . To project onto it, we need to remove all components that violate either of these conditions.



# THE PROJECTION ALGORITHM

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## Step 1 $J_z$ projection (via ancilla):

$$\begin{aligned} |\psi(t_i)\rangle &= \exp(-it_i J_z \otimes Y_a) |\psi\rangle \otimes |0\rangle_a \\ &= \cos(J_z t_i) |\psi\rangle \otimes |0\rangle_a + \sin(J_z t_i) |\psi\rangle \otimes |1\rangle_a \end{aligned}$$

Measuring the ancilla in  $|0\rangle$  projects onto  $J_z = 0$ . However,  $J > 0$  components with  $J_z = 0$  also pass through, so this step alone is not enough.

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## Step 2 $R_x$ rotation (system qubits only):

$$|\psi(t_i)\rangle = \exp(-it_i J_x) |\psi\rangle$$

The  $R_x$  rotation mixes  $J > 0$  components into states with different  $J_z$  values including non-zero  $J_z$ . The  $J = 0$  state is invariant under all rotations and is never affected.

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**Repeating this pair removes more and more of the unwanted  $J > 0$  components, until only  $J = 0, J_z = 0$  remains.**

# THE PROJECTION ALGORITHM

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1

## ENTANGLE WITH ANCILLA

The system evolves under  $\exp(-it_i J_z \otimes Y_a)$ , entangling it with the ancilla.

# THE PROJECTION ALGORITHM

---

1



2

**ENTANGLE  
WITH ANCILLA**

**MEASURE  
ANCILLA**

The system evolves under  $\exp(-it_j J_z \otimes Y_a)$ , entangling it with the ancilla.

Components with  $J_z = 0$  go to ancilla  $|0\rangle$ , components with  $J_z \neq 0$  go to ancilla  $|1\rangle$

# THE PROJECTION ALGORITHM

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**ROTATE  
BY  $R_X(\pi/2)$**

$R_x$  reshuffles  $J > 0$  components out, only  $J = 0$  passes through, because the  $J = 0$  state is invariant under all rotations and is never affected by  $R_x$

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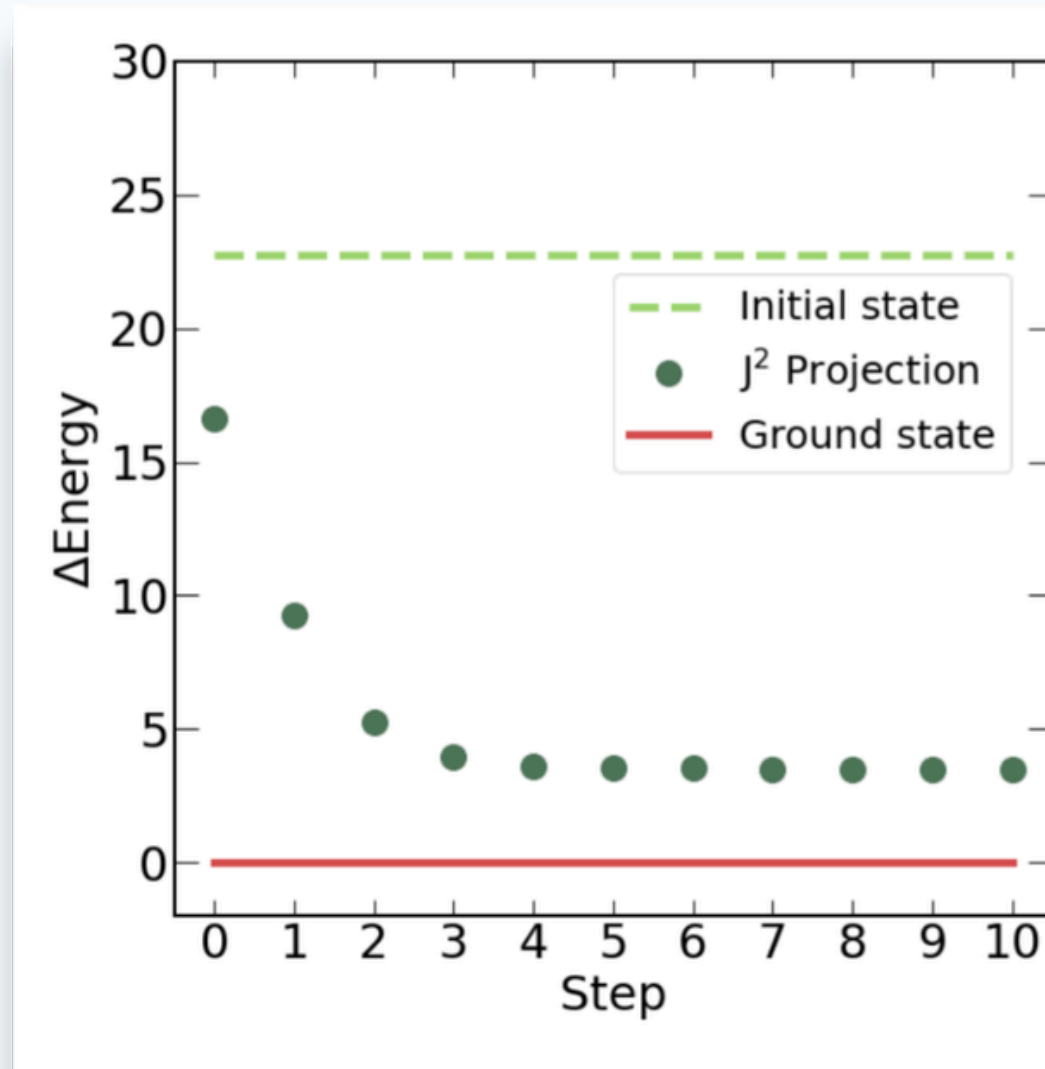


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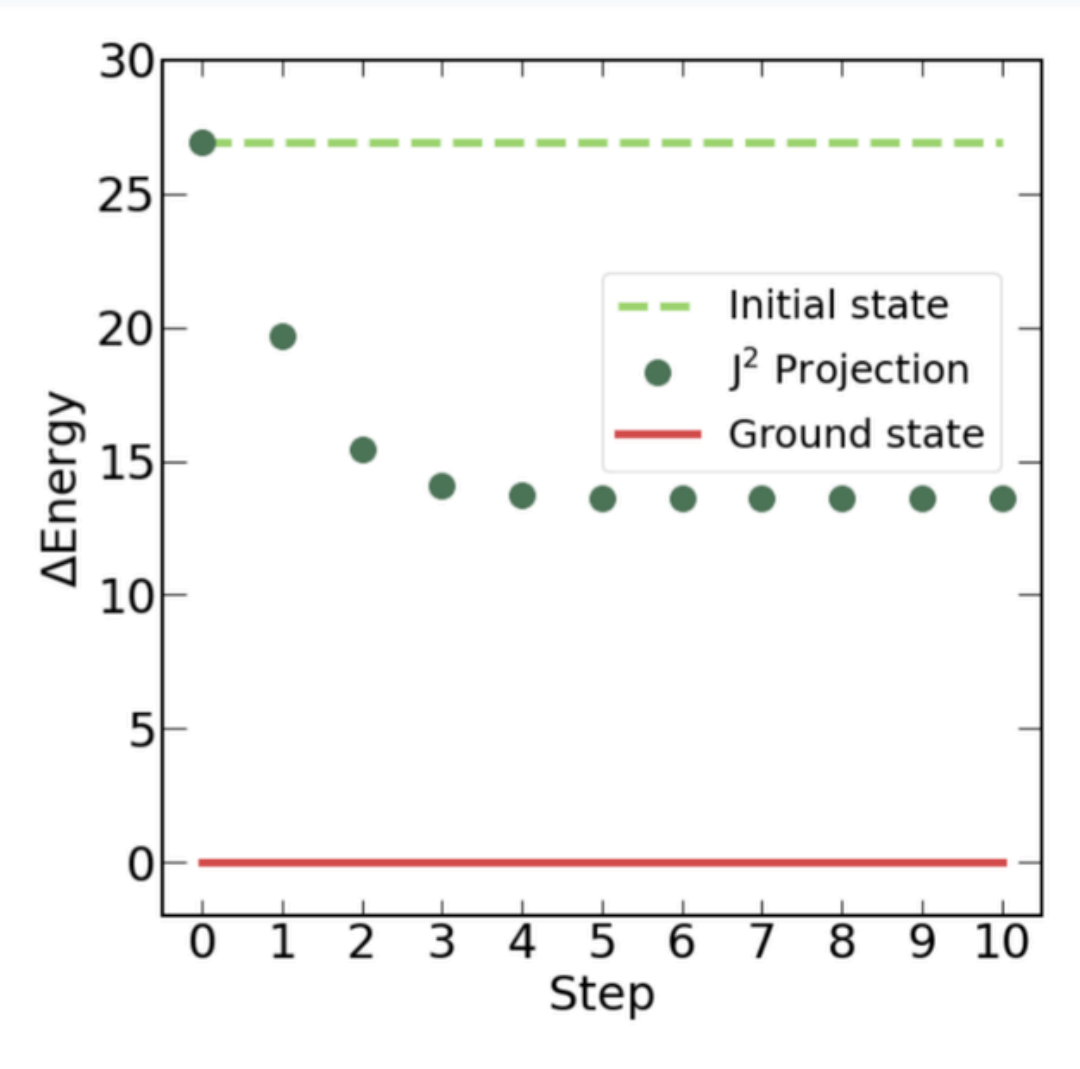
**REPEAT**

# PROJECTION RESULTS

## NEUTRINO MODEL



## HEISENBERG MODEL



- Heisenberg model: ~50% additional spatial symmetries still needed
- Neutrino model: ~80% of the way to exact ground state after 11 steps

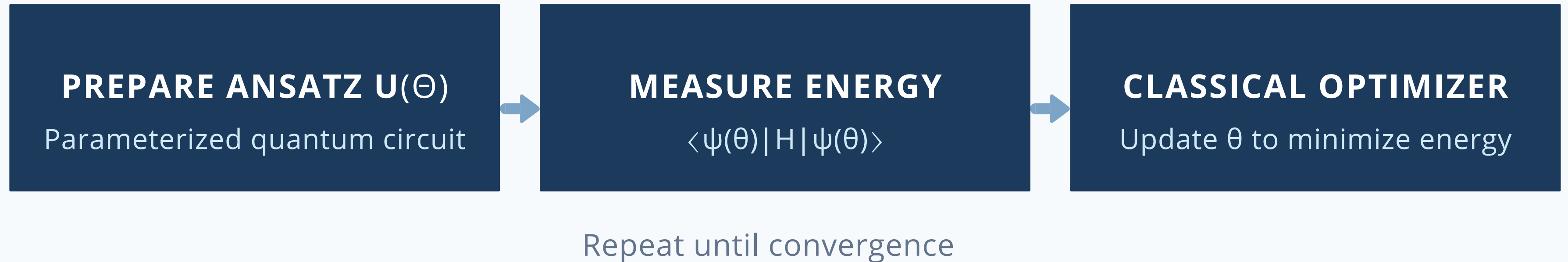
After projection    **Neutrino: fidelity 2.4% → 56.7%**    **Heisenberg: fidelity 8.5% → 29.6%**

# 4

## **RESULTS**

Comparison of performance with and without symmetries

# VARIATIONAL QUANTUM EIGENSOLVER



- **Variational principle:**  $\langle \psi | H | \psi \rangle \geq E_0$  for any trial state, minimizing energy gives best approximation
- **Main challenge:** barren plateaus; gradient becomes exponentially small, optimizer gets stuck
- **Solutions:** start close to the correct answer, reduce number of parameters
- Symmetry-projected initial state is an ideal warm start; fidelity already above **29–57%**

# VQE: WITHOUT VS. WITH SYMMETRY PROJECTION

## Without Projection

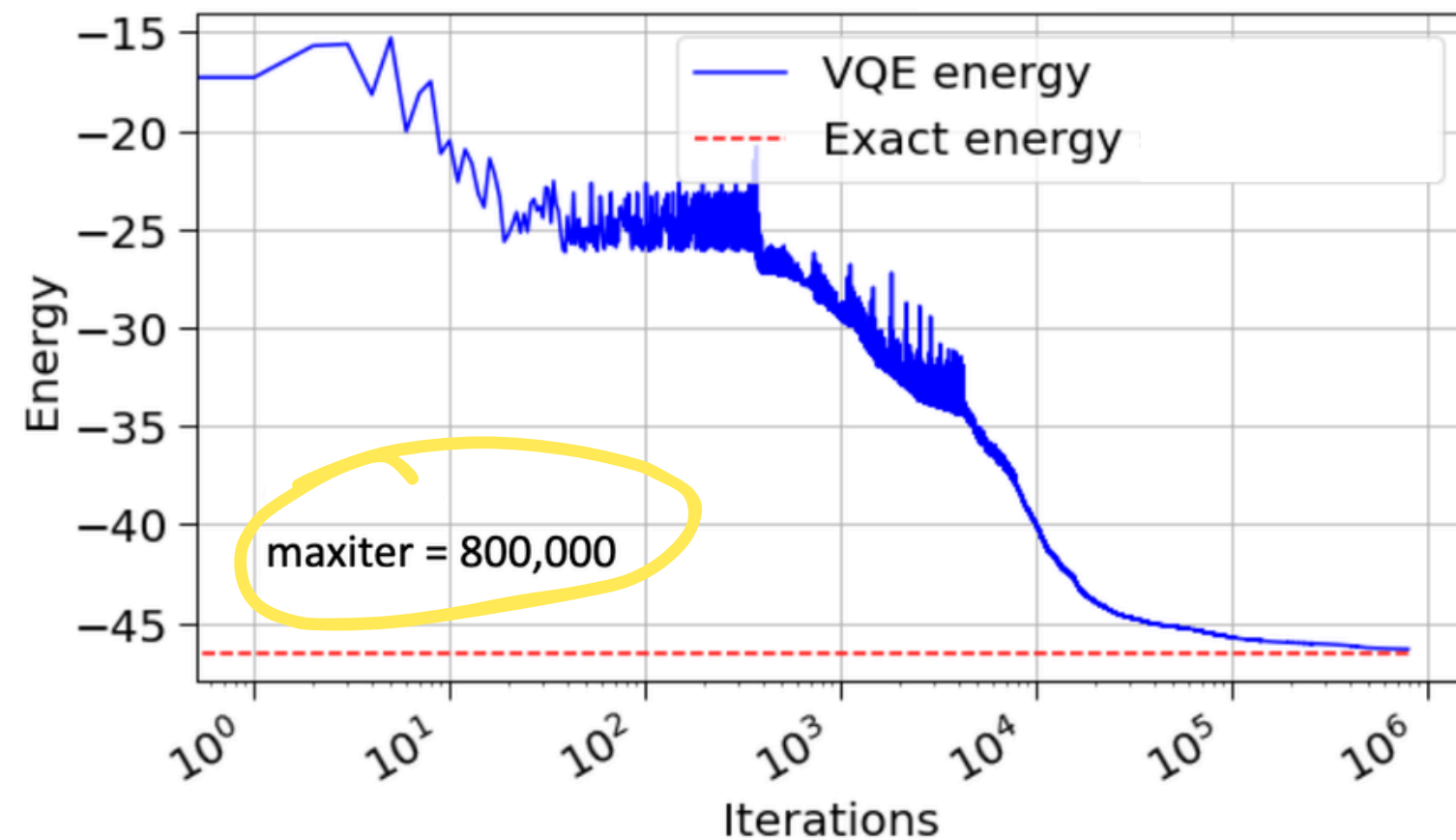
- Searches full 4096-dimensional Hilbert space
- No guidance on which subspace matters
- Prone to barren plateaus and local minima
- May converge to wrong symmetry sector
- Low final fidelity with exact ground state

## With Projection

- Searches only the correct symmetry subspace
- Starts from high-fidelity projected state
- Better-behaved optimization landscape
- Swap ansatz preserves  $J$  and  $J_z$  throughout

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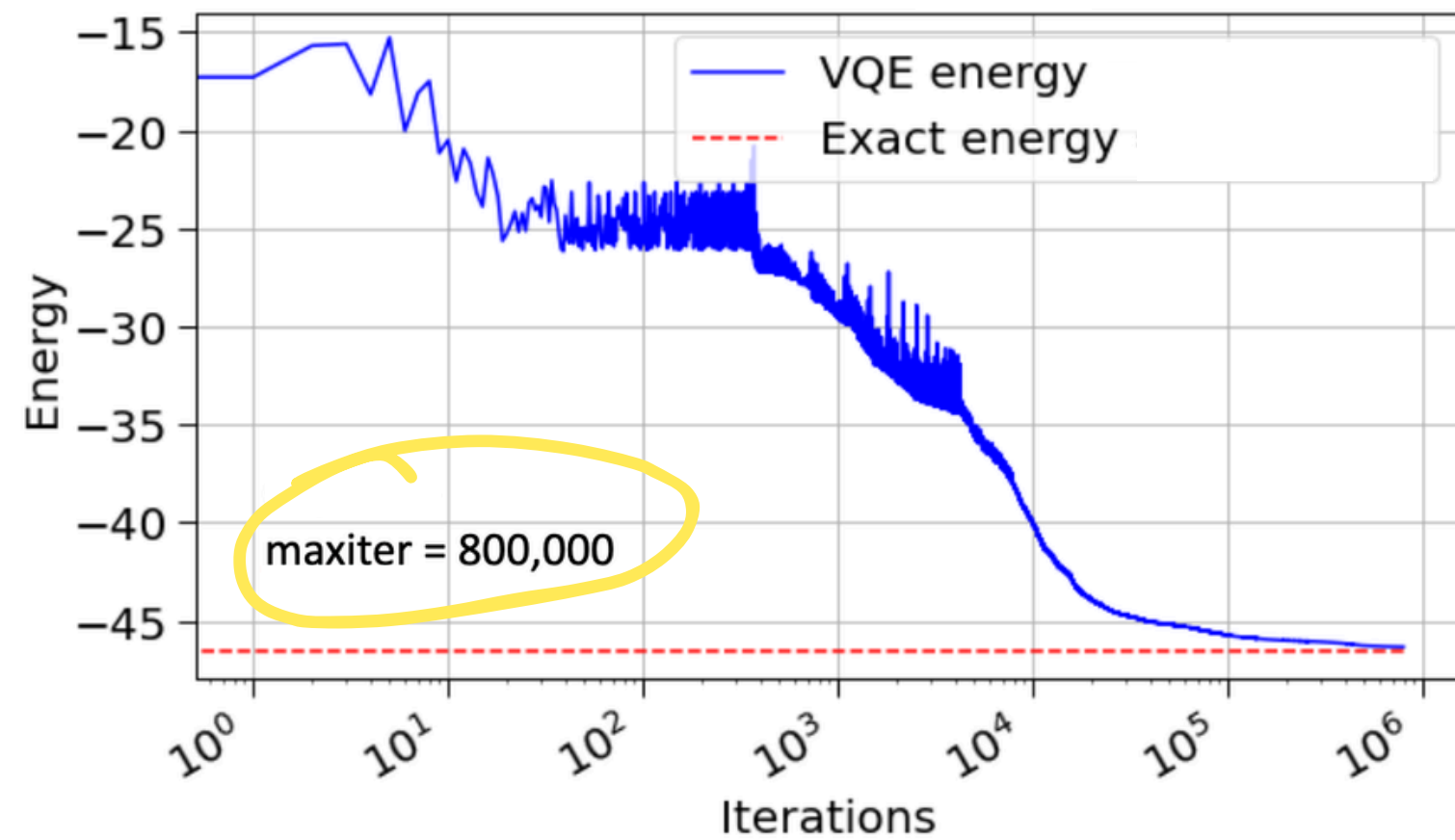


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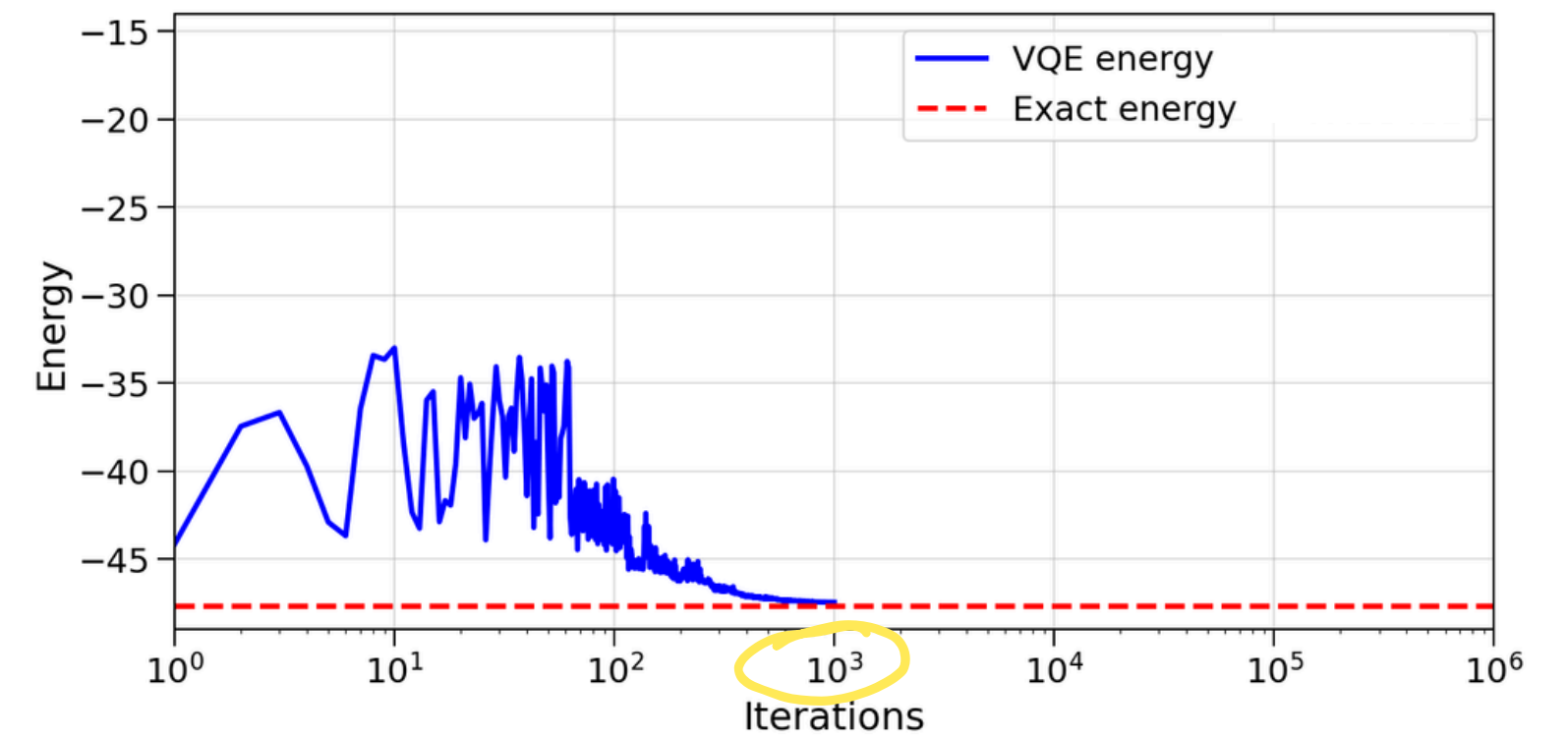
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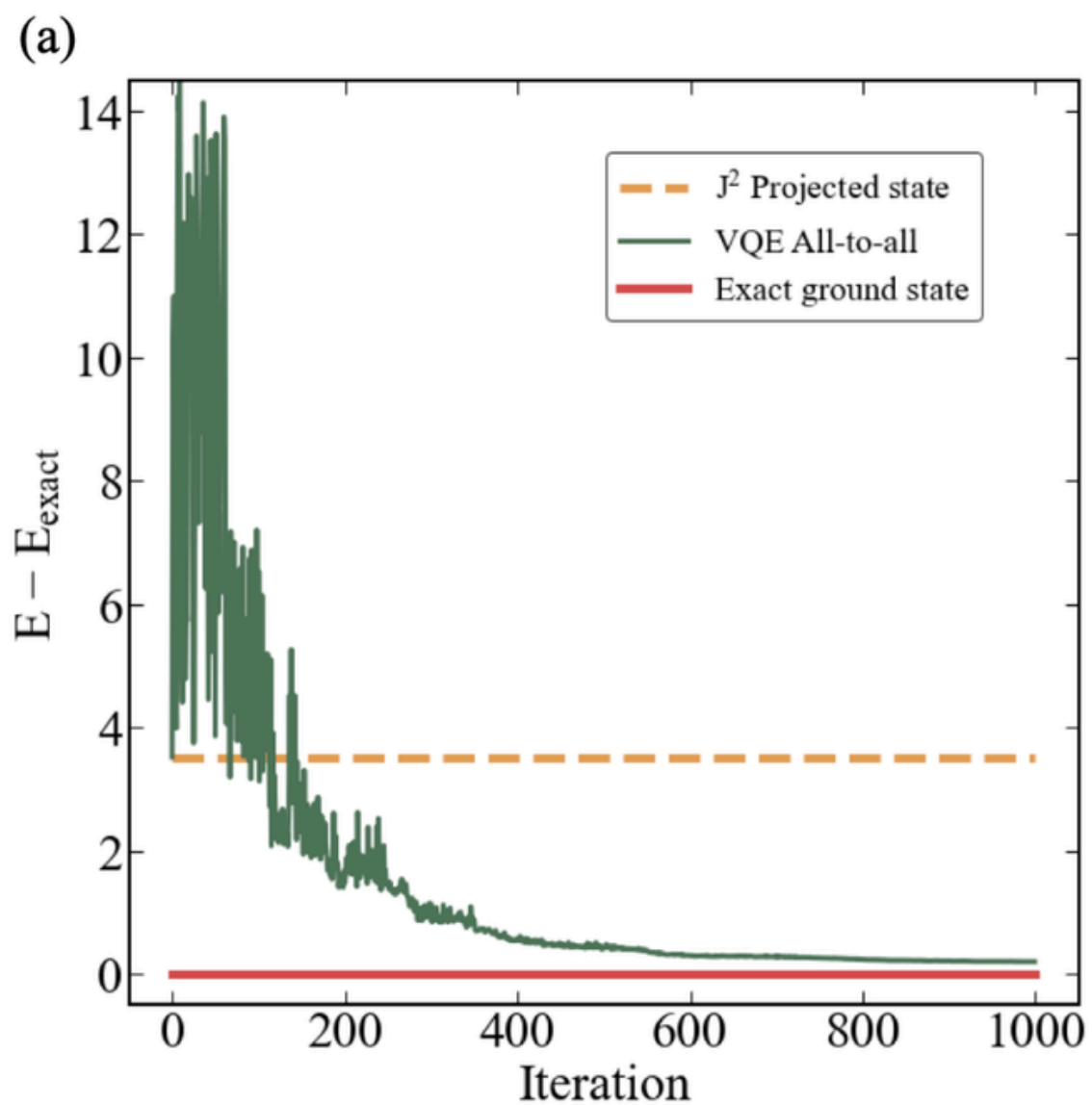


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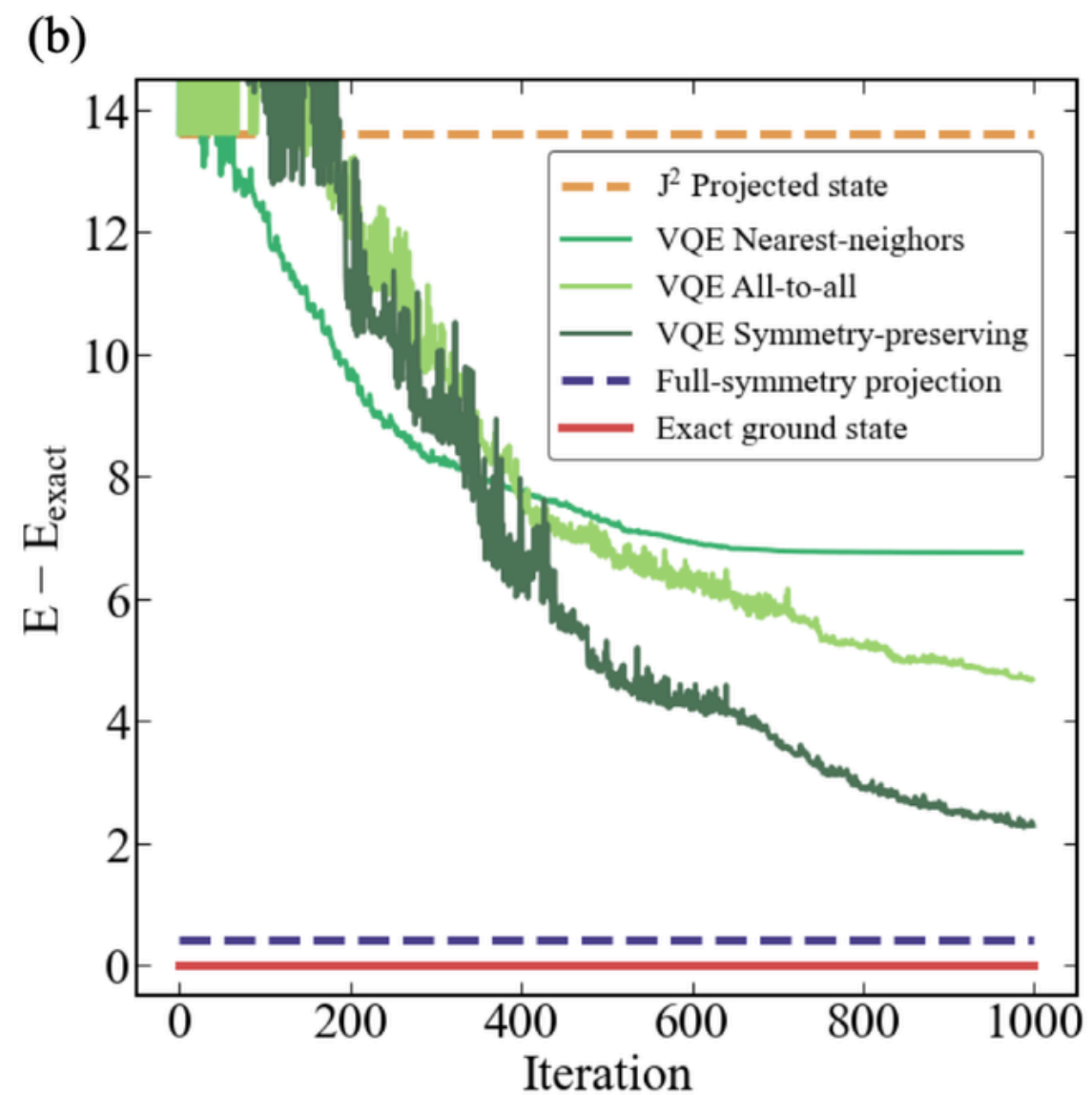


# RESULTS

## NEUTRINO MODEL



## HEISENBERG MODEL



# RESULTS

STEP	NEUTRINO FIDELITY	HEISENBERG FIDELITY
Initial product state	2.4%	8.5%
J <sup>2</sup> Projection (11 steps)	56.7%	29.6%
VQE convergence	<b>98.8%</b>	89.7%
Translation/mirror projection	—	<b>98.5%</b>

Both models reach >98% fidelity after the full symmetry pipeline.

# **FUTURE WORK**

Connecting to quantum phase estimation

# QUANTUM PHASE ESTIMATION

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QPE: controlled time evolution  $\exp\{-iHt\}$  + quantum Fourier transform  $\rightarrow$  extract eigenvalues (energies) directly.  
Precision scales with total evolution time.

- VQE gives a good approximation, QPE / IQPE give exact (or near-exact) energies
- **Success probability depends on overlap of initial state with ground state**
- IQPE (iterative QPE): lower qubit overhead, more practical for near-term devices

# CONCLUSIONS

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arxiv: 2510.06702

## **Symmetry:**

- Reduces the relevant Hilbert space, focusing the algorithm on physically relevant states only
- Improves the initial state dramatically, bringing it close to the ground state before any optimization
- Guides VQE to converge to the exact ground state reliably
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**THANK YOU FOR YOUR ATTENTION!**