

# Extensions and applications of admissible and attainable convergence theory for GMRES

Kirk M. Soodhalter

School of Mathematics  
Trinity College Dublin,  
<http://math.soodhalter.com>

18 May 2026

Partially supported by *Research Ireland* Grant 22/EPSRC/3857



**Trinity College Dublin**  
Coláiste na Tríonóide, Baile Átha Cliath  
The University of Dublin

Given  $A$  and  $\mathbf{b}$ , the  $m$ th Krylov subspace is defined

$$\mathcal{K}_m(A, \mathbf{b}) = \text{span} \{ \mathbf{b}, A\mathbf{b}, \dots, A^{m-1}\mathbf{b} \}.$$

Thus,  $\mathbf{u} \in \mathcal{K}_m(A, \mathbf{b})$  is such that

$$\mathbf{u} = p(A)\mathbf{b}$$

where  $p(x)$  is a polynomial of  $\deg p < m$ .

## Definition

The basis  $\{ \mathbf{b}, A\mathbf{b}, \dots, A^{m-1}\mathbf{b} \}$  is called a **Krylov basis** and a matrix with Krylov basis as columns is called a **Krylov matrix**,

$$K_m = [ \mathbf{b} \quad A\mathbf{b} \quad \dots \quad A^{m-1}\mathbf{b} ] \in \mathbb{R}^{n \times m}$$

We focus on solving linear systems:

$$A(\mathbf{x}_0 + \mathbf{t}) = \mathbf{b} \iff A\mathbf{t} = \mathbf{b} - A\mathbf{x}_0 = \mathbf{r}_0$$

- Choose  $\mathbf{x}_m = \mathbf{x}_0 + \mathbf{t}_m$ . Let  $\mathbf{r}_m = \mathbf{r}_0 - A\mathbf{t}_m$ .
- **GMRES**<sup>1</sup> :  $\mathbf{t}_m = \arg \min_{\mathbf{t} \in \mathcal{K}_m(A, \mathbf{r}_0)} \|\mathbf{r}_0 - A\mathbf{t}\|$   
→ This is equivalent to  $\mathbf{r}_m \perp A\mathcal{K}_m(A, \mathbf{r}_0)$
- **FOM**<sup>2</sup> :  $\mathbf{r}_m \perp \mathcal{K}_m(A, \mathbf{r}_0)$

---

<sup>1</sup>Generalized Minimum Residual Method

<sup>2</sup>Full Orthogonalization Method

# Basis construction for *optimal* Krylov methods

- The Krylov basis becomes numerically dependent rapidly
- The *Arnoldi process* iteratively builds an orthonormal basis<sup>3</sup>
- $V_m = [\mathbf{v}_1, \dots, \mathbf{v}_m] \in \mathbb{R}^{n \times m}$  has this basis as columns

## Arnoldi relation

$AV_m = V_{m+1}\underline{H}_m = V_m H_m + \alpha \mathbf{v}_{m+1} \mathbf{e}_m^T$  where  $\underline{H}_m \in \mathbb{R}^{(m+1) \times m}$  and  $H_m \in \mathbb{R}^{m \times m}$  are upper-Hessenberg.

<sup>a</sup> $\mathbf{e}_m$  is the  $m$ th canonical basis vector

- **GMRES**:  $\mathbf{y}_m = \arg \min_{\mathbf{y} \in \mathbb{R}^m} \|\underline{H}_m \mathbf{y} - \|\mathbf{r}_0\| \mathbf{e}_1\|_2$
- **FOM**:  $H_m \mathbf{y}_m = \|\mathbf{r}_0\| \mathbf{e}_1$

---

<sup>3</sup>A Gram-Schmidt-type process in which one orthogonalizes after each application of the operator  $A$





# Relevant Krylov convergence results

Krylov subspace projection methods are often used to estimate eigenvalues and eigenvectors

- **Ritz:**  $V_m^T (AV_m \mathbf{y} - \sigma V_m \mathbf{y}) = \mathbf{0}$  yields eigenpairs of  $H_m$

Implicit assumption: Ritz values quickly become good approximations to eigenvalues

# Admissible and attainable convergence of GMRES

# What governs convergence of Krylov iterative solvers?

- Decomposition of  $\mathbf{r}_0$  in the eigenbasis since

$$\mathbf{r}_m = Zq(\Lambda)Z^{-1}\mathbf{r}_0$$

- A normal – eigenvalues/interpolation govern general behavior, as indicated by the bound

$$\|\mathbf{r}_m\| \leq \|\mathbf{r}_0\| \min_{\substack{\deg q \leq m \\ q(0)=1}} \max_{i=1,2,\dots,n} |q(\lambda_i)|$$

- A non-normal – more complicated; eigenvalues seem to play a role, but what about eigenvectors?

$$\|\mathbf{r}_m\| \leq \|Z\| \|Z^{-1}\| \|\mathbf{r}_0\| \min_{\substack{\deg q \leq m \\ q(0)=1}} \max_{i=1,2,\dots,n} |q(\lambda_i)|$$



Eigenvalues strongly  
related to convergence



Eigenvalues completely  
un-related to convergence

This is an important question because often assumed true

- Seems to be the case for many problems
- Convergence theories and preconditioning strategies often based on eigenvalue distribution
- Finally, it was proven false; eigenvalues can be completely divorced from convergence behavior

SIAM J. MATRIX ANAL. APPL.  
Vol. 17, No. 3, pp. 465–469, July 1996

© 1996 Society for Industrial and Applied Mathematics  
001

## ANY NONINCREASING CONVERGENCE CURVE IS POSSIBLE FOR GMRES\*

ANNE GREENBAUM<sup>†</sup>, VLASTIMIL PTÁK<sup>‡</sup>, AND ZDENĚK STRAKOŠ<sup>‡</sup>

**Abstract.** Given a nonincreasing positive sequence  $f(0) \geq f(1) \geq \dots \geq f(n-1) > 0$ , it is shown that there exists an  $n$  by  $n$  matrix  $A$  and a vector  $r^0$  with  $\|r^0\| = f(0)$  such that  $f(k) = \|r^k\|$ ,  $k = 1, \dots, n-1$ , where  $r^k$  is the residual at step  $k$  of the GMRES algorithm applied to the linear system  $Ax = b$ , with initial residual  $r^0 = b - Ax^0$ . Moreover, the matrix  $A$  can be chosen to have any desired eigenvalues.

## ANY NONINCREASING CONVERGENCE CURVE IS POSSIBLE FOR GMRES\*

ANNE GREENBAUM<sup>†</sup>, VLASTIMIL PTÁK<sup>‡</sup>, AND ZDENĚK STRAKOŠ<sup>‡</sup>

**Abstract.** Given a nonincreasing positive sequence  $f(0) \geq f(1) \geq \dots \geq f(n-1) > 0$ , it is shown that there exists an  $n$  by  $n$  matrix  $A$  and a vector  $r^0$  with  $\|r^0\| = f(0)$  such that  $f(k) = \|r^k\|$ ,  $k = 1, \dots, n-1$ , where  $r^k$  is the residual at step  $k$  of the GMRES algorithm applied to the linear system  $Ax = b$ , with initial residual  $r^0 = b - Ax^0$ . Moreover, the matrix  $A$  can be chosen to have any desired eigenvalues.

- GMRES residual norms always non-increasing
- Given such a sequence  $f(0) \geq f(1) \geq \dots \geq f(n-1) > 0$ ,
- There exist  $A \in \mathbb{R}^{n \times n}$  and  $r_0$  such that GMRES for  $At = r_0$  has residuals with  $\|r_k\| = f(k)$  for all  $k$ .
- $A$  can be constructed to have any eigenvalues.

*Subsequent works refined results/language clarifying presentation*

## Aside: the friendly companion matrix

Recall that we can associate to the *characteristic polynomial*

$$p_A(z) = \det(zI - A) = z^n - \sum_{i=0}^{n-1} c_i z^i,$$

the companion matrix

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & c_0 \\ 1 & 0 & \dots & 0 & c_1 \\ 0 & 1 & \dots & 0 & c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & c_{n-1} \end{bmatrix}$$

$A$  and  $C$  have the same eigenvalues with multiplicity

# Going full Hessenberg (factorization)

- Arnoldi constructs partial orthogonal Hessenberg factorization of  $A$
- Carried to step  $n - 1$  generically produces full factorization  $A = HV^T$
- $H$  is similar to  $A$
- Krylov companion matrix relation ( $K_n =: K; V_n =: V; H_n =: H$ ):

$$AK = KC \iff A = KCK^{-1}$$

- QR-factorization:

$$K = \underbrace{V}_Q \underbrace{DU}_R \implies A = V \underbrace{DUCU^{-1}D^{-1}}_H V^*$$

# Scaled diagonal QR-factorization

We introduced the QR-factorization:  $K = \underbrace{V}_Q \underbrace{DU}_R$

- We scale the rows of  $R$  so that  $U$  has unit diagonal
- $D$  is diagonal and stores the row scalings
- $U$  is upper triangular with unit diagonal  $\implies$  so is  $U^{-1}$

$$U^{-1} = \begin{bmatrix} 1 & c_0^{(1)} & c_0^{(2)} & c_0^{(3)} & \cdots & c_0^{(n-1)} \\ 0 & 1 & c_1^{(2)} & c_1^{(3)} & \cdots & c_1^{(n-1)} \\ 0 & 0 & 1 & c_2^{(3)} & \cdots & c_2^{(n-1)} \\ 0 & 0 & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & c_{n-2}^{(n-1)} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Specifying convergence – $n^2 + n$ degrees of freedom

We frame as an argument about degrees of freedom (D.O.F.)

- $A$  has  $n^2$  D.O.F.
  - $n$  eigenvalues (with multiplicity)
  - $V$  ( $n$  orthonormal Arnoldi vectors) from orthogonal matrices with  $n(n-1)/2$  degrees of freedom
  - Ritz values at every iteration (except  $n$ -th)<sup>4</sup>:

$$1 + 2 + 3 + \cdots + n - 1 = n(n - 1)/2,$$

- $n + 2 \cdot n(n-1)/2 = n + n^2 - n = n^2$
- $r_0$  has  $n$  D.O.F.
  - an admissible sequence of  $n$  residual norms

---

<sup>4</sup>restriction: stagnation  $\iff$  zero Ritz values

# Any admissible convergence curve with any eigenvalues

Proving the theorem constructively by using the decomposition

$$A = VDUCU^{-1}D^{-1}V^*,$$

- Assign eigenvalues – via fixing coefficients  $c_i$  in  $C$
- Assign all Ritz values –  $m$ -th Ritz vals are eigenvalues of  $H_m$ 
  - coefficients  $\left\{c_{i-1}^{(m+1)}\right\}_{i=0}^{m-2}$  from  $U^{-1}$  defines char. poly. of  $H_m$
- $V$  can be chosen arbitrarily
- $\mathbf{r}_0 = VDU\mathbf{e}_1$  determined by  $n$  diagonal entries of  $D$

Remark (something is missing...)

*How do we use  $\mathbf{r}_0$  to specify convergence?*

# Specifying convergence by building $\mathbf{r}_0$

Representation  $\mathbf{r}_0 = VDU\mathbf{e}_1$  does not describe convergence

- Recall  $\mathbf{r}_m \perp AK_m(A, \mathbf{r}_0)$
- Let  $\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$  be orthonormal basis for  $\mathbb{R}^n$
- $\text{span}\{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\} = AK_m(A, \mathbf{r}_0)$  for all  $m \leq n$
- Projection method:  
$$\mathbf{r}_0 = \sum_{i=1}^n h_i \mathbf{w}_i \implies \mathbf{r}_m = \sum_{i=m+1}^n h_i \mathbf{w}_i$$
- Successive residual norm relation:  $|h_i|^2 = \|\mathbf{r}_{k-1}\|^2 - \|\mathbf{r}_k\|^2$

## Specify convergence implicitly

Use  $h_1, h_2, \dots, h_n$  as final  $n$  degrees of freedom; choose with “correct” absolute values

This construction is a special case of the *Frobenius normal form*

- $A = S\widehat{C}S^{-1}$ 
  - $S$  generated by a repeated multiplication of a vector by  $A$
  - If invariant subspace reached, continue with vector in complement
- $\widehat{C}$  block diagonal: companion matrices of minimal polynomial for each cyclic subspace
- If GMRES takes all  $n$  steps, normal form starting with  $r_0$  yields  $\widehat{C} = C$  and  $S = VDU$
- If invariant subspace reached, GMRES converges. Rest of normal form plays no role <sup>5</sup>

---

<sup>5</sup>corresponding to early convergence specification [Tebbens and Meurant 2014]

## KRYLOV SEQUENCES OF MAXIMAL LENGTH AND CONVERGENCE OF GMRES \*

M. ARIOLI<sup>1</sup>, V. PTÁK<sup>2</sup> and Z. STRAKOŠ<sup>2</sup> †

<sup>1</sup>IAN CNR, Via Abbiategrasso 209, 27100 Pavia, Italy  
email: arioli@dragon.ian.pv.cnr.it

<sup>2</sup>ICS AS CR, Pod Vodárenskou věží 2, 182 07 Praha 8, Czech Republic  
email: ptak@math.cas.cz, strakos@uivt.cas.cz

### Abstract.

In most practical cases, the convergence of the GMRES method applied to a linear algebraic system  $Az = b$  is determined by the distribution of eigenvalues of  $A$ . In theory, however, the information about the eigenvalues alone is not sufficient for determining the convergence. In this paper the previous work of Greenbaum et al. is extended in the following direction. It is given a complete parametrization of the set of all pairs  $\{A, b\}$  for which  $\text{GMRES}(A, b)$  generates the prescribed convergence curve while the matrix  $A$  has the prescribed eigenvalues. Moreover, a characterization of the right hand sides  $b$  for which the  $\text{GMRES}(A, b)$  converges exactly in  $m$  steps, where  $m$  is the degree of the minimal polynomial of  $A$ , is given.

- $AK_m(A, r_0)$  basis: parameterization  $A = WR CR^{-1}W^*$
- $R$  is upper-triangular
- Right-hand side  $r_0 = Wh$

Usefulness:

- Shows eigenvalues not fully descriptive for convergence
- Shows mechanics of GMRES convergence
- Ritz eigenvalue estimates can be arbitrarily bad

Limitations:

- Pathological cases unrelated to application problems
- Describes residual norm convergence result
  - errors still are connected to eigenvalues
- Construction is numerically unstable
  - non-trivial demonstrations difficult

# Extension to restarted GMRES

GMRES has practical computational constraints

- Arnoldi vectors may fill memory
- Computational cost increases with each iteration

## Mitigation: restarting

After  $m$  iterations, discard Arnoldi vectors and set

$$\mathbf{x}_0 \leftarrow \mathbf{x}_m = \mathbf{x}_0 + \mathbf{t}_m$$

$$\mathbf{r}_0 \leftarrow \mathbf{r}_m = \mathbf{b} - A\mathbf{x}_m$$

Thereafter apply another cycle of GMRES iterations. Restart as necessary until convergence.

Numerical Algorithms (2020) 84:1329–1352

<https://doi.org/10.1007/s11075-019-00846-z>

ORIGINAL PAPER

## On the residual norms, the Ritz values and the harmonic Ritz values that can be generated by restarted GMRES

Jurjen Duintjer Tebbens<sup>1,2</sup>  · Gérard Meurant<sup>3</sup>

Received: 24 July 2019 / Accepted: 5 November 2019 / Published online: 12 December 2019

© Springer Science+Business Media, LLC, part of Springer Nature 2019

- Each GMRES cycle specified using existing theory
- Link cycles together via passing final residual
- End-of-cycle stagnation restricts admissible convergence
- Assumes convergence at  $n$ th iteration

# Admissible convergence behavior for restarted GMRES

- Restarted GMRES norms always non-increasing
- Arbitrary non-increasing residual norms as long as

$$f_0^{(i-1)} \geq f_1^{(i-1)} \geq \dots \geq f_{m-1}^{(i-1)} > f_0^{(i)} \geq \dots \geq f_{m-1}^{(i)}$$

$i = 2, 3, \dots, \text{end (the cycle number)}$

- End-of-cycle stagnating iterations mirrored after restart,

$$f_0^{(i-1)} \geq \dots \geq f_{m-s}^{(i-1)} = \dots = f_{m-1}^{(i-1)} = f_0^{(i)} = \dots = f_s^{(i)} \geq \dots$$

→ New proofs of this and other results in [S. 2026]

# Building a restarted GMRES factorization

- Specifying restarted GMRES convergence history specifies  $A$
- Non-orthogonal basis of restarted Arnoldi vectors

$$\begin{aligned}\tilde{V} &= [V^{(1)} \quad V^{(2)} \quad \dots \quad V^{(\ell)}] \\ AV^{(k)} &= [V^{(k)} \quad V^{(k+1)}\mathbf{e}_1] \underline{\tilde{H}^{(k)}}\end{aligned}$$

- If final iteration specified to converge, then

$$A\tilde{V} = \tilde{V} \underbrace{\begin{bmatrix} \underline{\tilde{H}^{(1)}} & & & & \\ & \underline{\tilde{H}^{(2)}} & & & \\ & & \ddots & & \\ & & & \underline{\tilde{H}^{(\ell-1)}} & \\ & & & & H_{m \times m}^{(\ell)} \end{bmatrix}}_{=:\tilde{H}}$$

- Fully determines  $A$ , specifies eigenvalues and all Ritz values

# Specifying behavior of restarted GMRES at iteration $n$

- Restarted GMRES may not converge at  $n$ th iteration
- This can be specified at a cost: we cannot specify eigen- and Ritz values

## Lemma from [S. 2026]

If restarted GMRES is specified to not converge at iteration  $n$ , then it follows

$$A\tilde{V} = \tilde{V} \left( \underbrace{\tilde{H} + \mathbf{c}\mathbf{e}_n^T}_{=:\tilde{H}(\mathbf{c})} \right),$$

where for  $\mathbf{u} = \tilde{V}\mathbf{c} \neq \mathbf{0}$ ,  $AV^{(\ell)} = [V^{(\ell)} \quad \mathbf{u}] \underline{H^{(\ell)}}$ .

# Extension to block GMRES

## ADMISSIBLE AND ATTAINABLE CONVERGENCE BEHAVIOR OF BLOCK ARNOLDI AND GMRES\*

MARIE KUBÍNOVÁ<sup>†</sup> AND KIRK M. SOODHALTER<sup>‡</sup>

*The second author would like to dedicate this work to the late Richard Timoney, who was very generous with his time in explaining the basics of \*-algebras to an uninitiated new colleague*

**Abstract.** It is well-established that any nonincreasing convergence curve is possible for GMRES and a family of pairs  $(A, b)$  can be constructed for which GMRES exhibits a given convergence curve with  $A$  having arbitrary spectrum. No analogue of this result has been established for block GMRES, wherein multiple right-hand sides are considered. By reframing the problem as a single linear system over a ring of square matrices, we develop convergence results for block Arnoldi and block GMRES. In particular, we show what convergence behavior is admissible for block GMRES and how the matrices and right-hand sides producing any admissible behavior can be constructed. Moreover, we show that the convergence of the block Arnoldi method for eigenvalue approximation can be almost fully independent of the convergence of block GMRES for the same coefficient matrix and the same starting vectors.

- Extends case to solving  $AX = B \in \mathbb{C}^{n \times s}$
- BL-GMRES works on  $n \times s$  “vectors” with  $s \times s$  “scalars”
- Uses one-sided vector space with \*-algebra scalars<sup>6</sup>
- Assume W.L.O.G. that  $n = Ns$

---

<sup>6</sup>Following [Frommer et al 2017]

Salient points:

- positive scalars (normalizing quantities): upper-triangular matrices with positive diagonal
- normalization: economy QR-factorization  $\mathbf{Z} = \mathbf{V}\mathbf{R}$
- block norm:  $|||\mathbf{Z}||| := \mathbf{R}$
- block inner product:  $\mathbf{V}, \mathbf{W} \in \mathbb{S}^N$ :  $\langle\langle \mathbf{V}, \mathbf{W} \rangle\rangle := \mathbf{W}^* \mathbf{V}$ .
- absolute value:  $|\mathbf{S}| := \text{CHOLU}(\mathbf{S}^* \mathbf{S})$
- normalizing quantities are partially ordered i. e.,

$$R_1 \leq R_2 \iff R_1^* R_1 \preceq R_2^* R_2,$$

This perspective allows for definition of admissible behavior.

## Admissible BL-GMRES convergence [Kubínová, S. 2020]

A sequence of normalizing quantities  $\{F_i\}$  is admissible if and only if

$$F_0 \geq F_1 \geq \dots \geq F_{n-1}$$

- We can construct  $A, B$  attaining this behavior via

$$A = \mathbf{V} \mathbf{D} \mathbf{U} \mathbf{C} \mathbf{U}^{-1} \mathbf{D}^{-1} \mathbf{V}^* \quad \text{and} \quad B = \mathbf{V} \mathbf{E}_1 F_0,$$

- Allows for specification of eigenvalues, Ritz values (usually) independent of convergence

# Highlights of this construction

$$A = \mathbf{V} \underbrace{DUCU^{-1}D^{-1}}_{\mathcal{H}} \mathbf{V}^* \quad \text{and} \quad B = \mathbf{V}E_1F_0,$$

- $\mathcal{H}$  - block upper Hessenberg
- Block companion matrix allows assignment of eigenvalues

$$C = \begin{bmatrix} 0 & 0 & \dots & 0 & C_0 \\ I & 0 & \dots & 0 & C_1 \\ 0 & I & \dots & 0 & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & C_{n-1} \end{bmatrix},$$

- $U^{-1}$  - block upper triangular, unit diagonal, allows for assignment of Ritz values
- Stagnation restrictions freedom in assigning Ritz values

- *Totally stagnation*:  $\mathbf{X}_m = \mathbf{X}_{m-1}$
- *Partial stagnation* w.r.t.  $\mathbf{u}$ :  $\mathbf{X}_m \neq \mathbf{X}_{m-1}$  but  $\mathbf{X}_m \mathbf{u} = \mathbf{X}_{m-1} \mathbf{u}$

## BL-GMRES stagnation and residual partial ordering [S. 2026]

There is no stagnation w.r.t. to any direction if and only if the residuals satisfy

$$|||\mathbf{R}_m||| < |||\mathbf{R}_{m-1}|||$$

# Extension to restarted BL-GMRES enabled by $*$ -algebra framework

This work comes from [S. 2026, *Numer. Alg.*]

- Each BL-GMRES cycle specified using existing theory
- Link cycles together via passing final block residual
- End-of-cycle stagnation restricts admissible convergence
- Begin by assuming convergence at  $n$ th iteration, then relax this restriction

- Restarted BL-GMRES block norms always “non-increasing”
- Arbitrary non-increasing residual norms as long as

$$F_0^{(i-1)} \geq F_1^{(i-1)} \geq \dots \geq F_{m-1}^{(i-1)} > F_0^{(i)} \geq \dots \geq F_{m-1}^{(i)}$$

$i = 2, 3, \dots$ , end (the cycle number)

- End-of-cycle stagnating iterations w.r.t.  $\mathbf{u}$  mirrored after restart,

$$\begin{aligned} \mathbf{u}^* \left\langle \left\langle \mathbf{R}_{m-s}^{(k)}, \mathbf{R}_{m-s}^{(k)} \right\rangle \right\rangle \mathbf{u} &= \mathbf{u}^* \left\langle \left\langle \mathbf{R}_m^{(k)}, \mathbf{R}_m^{(k)} \right\rangle \right\rangle \mathbf{u} \\ \implies \mathbf{u}^* \left\langle \left\langle \mathbf{R}_0^{(k+1)}, \mathbf{R}_0^{(k+1)} \right\rangle \right\rangle \mathbf{u} &= \mathbf{u}^* \left\langle \left\langle \mathbf{R}_s^{(k+1)}, \mathbf{R}_s^{(k+1)} \right\rangle \right\rangle \mathbf{u}. \end{aligned}$$

# Building a restarted BL-GMRES factorization

- Specifying restarted BL-GMRES convergence history specifies  $A$
- Non-orthogonal basis of restarted Arnoldi vectors

$$\begin{aligned}\tilde{\mathbf{V}} &= [\mathbf{v}^{(1)} \quad \mathbf{v}^{(2)} \quad \dots \quad \mathbf{v}^{(\ell)}] \\ A\mathbf{v}^{(k)} &= [\mathbf{v}^{(k)} \quad \mathbf{v}^{(k+1)} \mathbf{E}_1] \underline{\tilde{\mathcal{H}}^{(k)}}\end{aligned}$$

- If final iteration specified to converge, then

$$A\tilde{\mathbf{V}} = \tilde{\mathbf{V}} \underbrace{\begin{bmatrix} \underline{\tilde{\mathcal{H}}^{(1)}} & & & & \\ & \underline{\tilde{\mathcal{H}}^{(2)}} & & & \\ & & \ddots & & \\ & & & \underline{\tilde{\mathcal{H}}^{(\ell-1)}} & \\ & & & & \mathcal{H}_{M \times M}^{(\ell)} \end{bmatrix}}_{=:\tilde{\mathcal{H}}}$$

- Fully determines  $A$ , specifies eigenvalues and all Ritz values

# Specifying non-convergence of at iteration $N$

- Restarted BL-GMRES may not converge at  $N$ th iteration
- This can be specified at a cost: we cannot specify eigen- and Ritz values

Lemma from [S. 2026]

If restarted BL-GMRES is specified to not converge at iteration  $N$ , then it follows

$$A\tilde{\mathbf{V}} = \tilde{\mathbf{V}} \left( \underbrace{\tilde{\mathcal{H}} + \mathbf{C}\mathbf{E}_N^T}_{=:\tilde{\mathcal{H}}(\mathbf{C})} \right),$$

where for  $\mathbf{U} = \tilde{\mathbf{V}}\mathbf{C} \neq \mathbf{0}$ ,  $A\mathbf{V}^{(\ell)} = [\mathbf{V}^{(\ell)} \quad \mathbf{U}] \underline{\mathcal{H}}^{(\ell)}$ .

# Flipping the script

Using the framework to better understand  
GMRES convergence for Toeplitz systems

Joint work with Jerry Chen (Univ. Galway)

# Toeplitz degrees of freedom and spectrum

$$B_n = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix},$$

We have  $\underbrace{2n-1}_{B_n} + \underbrace{n}_{\text{R-H.S}}$  D.O.F. Unanswered questions:

- Can we to assign GMRES behavior? Spectral quantities?
- How determinative are eigenvalues for convergence?
- Are there families exhibiting the same convergence?

Much work done in this direction

- **Toeplitz symbol**; e. g. Garoni and Serra-Capizzano and refs therein
- **Field of values analysis**; e. g. [Abu-Labdeh and Pestana 2025]
- **Pseudo-spectral analysis**; e. g. Trefethen and Embree and refs therein
- **Eigenvalue clustering/grouping**; e. g. [Campbell et al 1996]
- **Eigenvector conditioning**; e. g. [Liesen and Strakoš 2004]
- **Best-case/worst-case behavior characterization**; e. g. [Li and Zhang 2009]

# Much work considers tridiagonal case

$$\begin{bmatrix} a & c & & & & \\ b & a & c & & & \\ & b & a & c & & \\ & & \ddots & \ddots & \ddots & \\ & & & b & a & c \\ & & & & b & a & c \\ & & & & & b & a \end{bmatrix} \mathbf{t} = \mathbf{r}_0$$

- 3 D.O.F. in  $A$  and  $n$  D.O.F. in  $\mathbf{r}_0$
- eigenvalues and eigenvectors in terms of triplet  $(a, b, c)$ :

$$\underbrace{\lambda_i = a + 2\sqrt{bc} \cos\left(\frac{i\pi}{n+1}\right)}_{2 \text{ D.O.F.}} \quad \text{and} \quad \underbrace{\mathbf{z}_i = \left( (b/c)^{j/2} \sin\left(\frac{ij\pi}{n+1}\right) \right)_j}_{1 \text{ D.O.F.}}$$

# What leeway do we have?

- Two D.O.F. determining eigenvalues (in a constrained way).
- One D.O.F. determining eigenvectors (and thus normality)
- Eigenvalue/vector specification is highly proscribed
- Specifying  $\{\|\mathbf{r}_i\|\}_i$  still requires  $n$  D.O.F.

## Tools at hand

- $A$  is tridiagonal and Toeplitz
- $WR = [A\mathbf{r}_0 \quad A^2\mathbf{r}_0 \quad \cdots \quad A^n\mathbf{r}_0]$
- The APS-parameterization  $A = WRCR^{-1}W^*$
- Residual convergence specified via  $\mathbf{r}_0 = W\mathbf{h}$

# Investigating various cases

- Given arbitrary unitary  $W$  and  $\mathbf{r}_0$ 
  - There almost never  $R$  exists yielding tridiagonal Toeplitz  $A$
- Given  $W$  generated from  $(A, \mathbf{r}_0)$  generating a convergence curve
  - For  $n > 4$ , there *does not exist* another distinct pair  $(\tilde{A}, \tilde{\mathbf{r}}_0)$  generating the same behavior
  - For  $A$ , any  $\tilde{\mathbf{r}}_0 = \sum_{i=1}^n \tilde{h}_i \mathbf{w}_i$  such that  $|\tilde{h}_i| = |h_i|$  produces the same convergence behavior

## Conclusion

Since spectral information is determined by  $(a, b, c)$ , these are coupled to the behavior of GMRES applied to the tridiagonal Toeplitz system

This framework does not just characterize pathological cases. It describe the mechanics of GMRES behavior for all GMRES situations

**Toeplitz convergence:** Long-term goal: characterize convergence patterns for more complicated structured problems

- Extend to Toeplitz with any bandwidth
- Block Toeplitz
- Toeplitz-Block Toeplitz
- Low-rank modifications of Toeplitz
- Generalized locally Toeplitz

**BL-GMRES convergence:** Extended to construct cases for which block Krylov subspace loses dimension

Thank you for your attention! Any  
questions/comments?

- [1] Fei Chen and Kirk M. Soodhalter. *Admissible convergence patterns for GMRES applied to tridiagonal Toeplitz systems*. In preparation. 2026.
- [2] Marie Kubínová and Kirk M. Soodhalter. “Admissible and Attainable Convergence Behavior of Block Arnoldi and GMRES”. In: *SIAM Journal on Matrix Analysis and Applications* 41.2 (Jan. 2020), pp. 464–486. ISSN: 1095-7162. DOI: 10.1137/19m1272469.
- [3] Kirk M Soodhalter. “Admissible and attainable convergence behavior with stagnation mirroring in restarted (block) GMRES”. In: *Numerical Algorithms* (Apr. 2026). DOI: 10.1007/s11075-026-02368-x.