Space-time tensor-product finite element methods for parabolic problems

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Model problem

$$\partial_t u - \Delta_x u = f \text{ in } Q := \Omega \times (0,T), \; u = 0 \text{ on } \Sigma = \partial\Omega \times (0,T), \; u(0) = u_0 \text{ in } \Omega$$

Variational formulation:

[OS 2015; Schwab, Stevenson 2009; Urban, Patera 2014; Andreev 2013, Mollet 2014; ...]

Find $u \in X = \{u \in Y : \partial_t u \in Y^*\}$, $u(0) = u_0$ in Ω:

$$b(u,v) := \langle \partial_t u, v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} = \langle f, v \rangle_Q$$

is satisfied for all $v \in Y = L^2(0,T; H^1_0(\Omega))$.

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is satisfied for all $v \in Y = L^2(0,T; H_0^1(\Omega))$.

Assumptions

- $f \in Y^*$, $u_0 \in L^2(\Omega)$, $\widetilde{u}_0 \in X \to u \in X$
- $f \in L^2(Q)$, $u_0 \in H_0^1(\Omega) \to u \in X \cap H^{2,1}(Q)$
- But what about discontinuous initial data, $u_0 \in H^s(\Omega)$, $s < 1/2$.

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Homogenization: Find $\widetilde{u} \in X_0 = \{w \in X : w(0) = 0\}$ such that

$$b(\widetilde{u}, v) = \langle f, v \rangle_Q - b(\widetilde{u}_0, v) \quad \text{for all } v \in Y.$$

Space-time FEM: Find $\widetilde{u}_h \in X_{0,h}$:

$$b(\widetilde{u}_h, v_h) = \langle f, v_h \rangle_Q - b(\widetilde{u}_0, v_h) \quad \text{for all } v_h \in Y_h.$$

Space-time finite element spaces

$$X_{0,h} \subset X_0, \quad Y_h \subset Y, \quad \dim X_{0,h} = \dim Y_h, \quad X_{0,h} \subset Y$$

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Discrete inf-sup stability condition

$$\|u_h\|_{X_{0,h}} \leq \sup_{0 \neq v_h \in Y_h} \frac{b(u_h, v_h)}{\|v_h\|_Y} \quad \text{for all } u_h \in X_{0,h}$$

Discrete norm $(u \in X_{0,}, w_h, v_h \in Y_h)$

$$\|u\|^2_{X_{0,h}} = \|u\|^2_Y + \|w_h\|^2_Y, \quad \langle \nabla_x w_h, \nabla_x v_h \rangle_{L^2(Q)} = \langle \partial_t u, v_h \rangle_Q$$

Proof of inf-sup condition

$$\overline{v}_h := u_h + w_h \in Y_h \quad b(u_h, \overline{v}_h) = \|\overline{v}_h\|^2_Y \geq \|\overline{v}_h\|_Y \|u_h\|_{X_{0,h}}$$

Space-time piecewise linear finite element space [OS: CMAM 2015]

$$Y_h = X_{0,h} = S^1_h(Q) \cap X_0$$

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- Adaptive least-squares space-time FEM [Köthe, Löscher, OS: arXiv 2023]
- Convection-diffusion problems [Köthe, OS: CMAM 2026]
- Rotating electric machines [Gangl, Gobrial, OS: CMAM 2025]
- Optimal control problems
  [Langer, OS, Tröltzsch, Yang: SISC/SINUM 2021; Langer, OS, Yang: ACOM 2024]

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Space-time tensor product finite element spaces

$$V_{h_x} = S_h^1(\Omega) \cap H_0^1(\Omega) = \text{span}\{\varphi_k\}_{k=1}^{M_x}, \; W_{h_t}^1 = S_h^1(0,T) \cap H_{0,}^1(0,T) = \text{span}\{\psi_i^1\}_{i=1}^{N_t}$$

Then,

$$X_{0,h} = W_{h_t}^1 \otimes V_{h_x},$$

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Then,

$$X_{0,h} = W_{h_t}^1 \otimes V_{h_x}, \quad Y_h = W_{h_t}^0 \otimes V_{h_x}, \quad W_{h_t}^0 = \text{span}\{\psi_i^0\}_{i=1}^{N_t}$$

Note that

$$X_{0,h} \not\subset Y_h, \quad \text{but} \quad \dim X_{0,h} = \dim Y_h = N_t \cdot M_x, \quad \partial_t u_h \in Y_h \text{ for } u_h \in X_{0,h}$$

Unique solvability follows from the discrete inf-sup condition

$$\|\partial_t u_h\|_{L^2(Q)} \leq \sup_{0 \neq v_h \in V_h} \frac{b(u_h, v_h)}{\|v_h\|_{L^2(Q)}}, \quad \overline{v}_h = \partial_t u_h \in Y_h, \; u_h \in X_{0,h}$$

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Discrete norm

$$\| u \|_X^2 := \; c_Q^{-2} \, \|\partial_t u\|_Y^2 + \|Q_{h_t}^0 u\|_Y^2, \quad \|\nabla_x Q_{h_x}^1 u\|_{L^2(\Omega)} \; \leq \; c_Q \, \|\nabla_x u\|_{L^2(\Omega)}$$

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Cea's lemma

$$\| \widetilde{u} - \widetilde{u}_h \|_X \leq \; \sqrt{2} \inf_{z_h \in X_{0,h}} \|\widetilde{u} - z_h\|_X$$

Approximation properties

$s \in [1,2], \sigma \in (\frac{1}{2}, 2]$:

$$\|\widetilde{u} - Q_{h_x}^1 I_{h_t}^1 \widetilde{u}\|_Y \leq c_1 \, h_x^{s-1} \, |\widetilde{u}|_{L^2(0,T;H^s(\Omega))} + c_2 \, h_t^\sigma \, |\widetilde{u}|_{H^\sigma(0,T;H_0^1(\Omega))}$$

$\tau \in [0,1], \varrho \in [1,2]$:

$$\|\partial_t(\widetilde{u} - Q_{h_x}^1 I_{h_t}^1 \widetilde{u})\|_{Y^*} \leq c_1 \, h_x^{1-\tau} \, \|\partial_t \widetilde{u}\|_{L^2(0,T;H^{-\tau}(\Omega))} + c_2 \, h_t^{\varrho-1} \, \|\widetilde{u}\|_{H^\varrho(0,T;H^{-1}(\Omega))}$$

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$s = 2, \; \sigma = 1, \; \tau = 0, \; \varrho = 2, \; h_t = h_x$:

$$\begin{aligned} & \|\!| \, \widetilde{u} - \widetilde{u}_h \, |\!\|_X \\ & \leq c \, h_x \left[|\widetilde{u}|^2_{L^2(0,T;H^2(\Omega))} + |\widetilde{u}|^2_{H^1(0,T;H^1_0(\Omega))} + \|\partial_t \widetilde{u}\|^2_{L^2(Q)} + \|\widetilde{u}\|^2_{H^2(0,T;H^{-1}(\Omega))} \right]^{1/2} \end{aligned}$$

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Error estimate in $L^2(Q)$:

$$\|\widetilde{u} - \widetilde{u}_h\|_{L^2(Q)} \, \leq \, c \, h_x^2 \left[\|\widetilde{u}\|_{H^1(0,T;H^2(\Omega))} \, + \, \|\widetilde{u}\|_{H^2(0,T;H^2(\Omega))} \right] .$$

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Error estimate in Y:

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Error estimate in Y:

$$\|\widetilde{u} - \widetilde{u}_h\|_Y \, \leq \, c(\widetilde{u}) \, h_x$$

Recall that we solve for $\widetilde{u}_h$ satisfying

$$b(\widetilde{u}_h + \widetilde{u}_0, v_h) = 0 \quad \text{for all } v_h \in Y_h.$$

But from a practical point of view, we have to solve for $\widehat{u}_h$ satisfying

$$b(\widehat{u}_h + Q^1_{h_x} I^1_{h_t} \widetilde{u}_0, v_h) = 0 \quad \text{for all } v_h \in Y_h,$$

and doing some reordering as known from inhomogeneous Dirichlet boundary conditions; application of Strang lemma.

Numerical example: smooth solution

$$\Omega = (-1,1), \quad T = 1, \quad h_t = h_x, \quad u(x,t) = \cos \pi t \, \sin \pi x$$

N_x	N_t	$\|u - u_h\|_{L^2(Q)}$	eoc	$\|u - u_h\|_Y$	eoc
2	2	$9.735 \cdot 10^{-1}$		$3.147 \cdot 10^{0}$	
4	4	$1.975 \cdot 10^{-1}$	2.30	$1.500 \cdot 10^{0}$	1.07
8	8	$4.360 \cdot 10^{-2}$	2.18	$7.141 \cdot 10^{-1}$	1.07
16	16	$1.082 \cdot 10^{-2}$	2.01	$3.562 \cdot 10^{-1}$	1.00
32	32	$2.704 \cdot 10^{-3}$	2.00	$1.781 \cdot 10^{-1}$	1.00
64	64	$6.760 \cdot 10^{-4}$	2.00	$8.904 \cdot 10^{-2}$	1.00

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All the previous error estimates are not applicable when considering less regular data, i.e., discontinuous initial conditions.

Discrete norm

$$\|\!| u \|\!|_X^2 = c_Q^{-2} \, \|\partial_t u\|_{Y^*}^2 + \|Q_{h_t}^0 u\|_Y^2 \leq c_Q^{-2} \, \|\partial_t u\|_{Y^*}^2 + \|u\|_Y^2 \leq c \, \|u\|_X^2$$

Norm equivalence [Andreev 2012]

$$c \, \min\{1, h_t^{-1} \, h_x^2\} \, \|u_h\|_X \leq \|\!| u_h \|\!|_X \quad \text{for all } u_h \in X_{0,h}$$

Discrete inf-sup condition

$$c \, \min\{1, h_t^{-1} \, h_x^2\} \, \|u_h\|_X \leq \sup_{0 \neq v_h \in Y_h} \frac{b(u_h, v_h)}{\|v_h\|_Y}$$

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Parabolic scaling

$$h_t = h_x^2$$

Cea's lemma

$$\|\widetilde{u} - \widetilde{u}_h\|_X \leq c \min_{z_h \in X_{0,h}} \|\widetilde{u} - z_h\|_X$$

Approximation properties

$s \in [1,2], \sigma \in (\frac{1}{2}, 2]$:

$$\|\widetilde{u} - Q_{h_x}^1 I_{h_t}^1 \widetilde{u}\|_Y \leq c_1 \, h_x^{s-1} \, |\widetilde{u}|_{L^2(0,T;H^s(\Omega))} + c_2 \, h_x^{-1} \, h_t^\sigma \, |\widetilde{u}|_{H^\sigma(0,T;L^2(\Omega))}$$

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$s = 2, \ \sigma = 1, \ \tau = 0, \ \varrho = \frac{3}{2}, \ h_t = h_x^2, \ f \equiv 0$:

$$\|\widetilde{u} - \widetilde{u}_h\|_X \leq c \, h_x \left[|\widetilde{u}|_{H^{2,1}(Q)} + \|\widetilde{u}\|_{H^{3/2}(0,T;H^{-1}(\Omega))} \right]$$

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With

$$\|\widetilde{u} - \widetilde{u}_h\|_X \leq c\, \|\widetilde{u}\|_X \leq c\, \|u_0\|_{L^2(\Omega)}$$

we finally conclude

$$\|\widetilde{u} - \widetilde{u}_h\|_X \leq c\, h_x^s\, \|u_0\|_{\widetilde{H}^s(\Omega)} \quad \text{for } u_0 \in \widetilde{H}^s(\Omega) = [L^2(\Omega), H_0^1(\Omega)]_{|s}, s \in [0,1]$$

Nitsche trick ($h_t = h_x^2$)

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$$\|\widetilde{u} - \widetilde{u}_h\|_{L^2(Q)} \leq c\, h_x \, \|\widetilde{u} - \widetilde{u}_h\|_X$$

Numerical example: singular solution

$$\Omega = (-1,1), \quad T = 1, \quad h_t = h_x^2, \quad f(x,t) \equiv 0, \quad u_0(x) = \begin{cases} 1 & \text{for } |x| < \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

N_x	N_t	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_Y$	eoc
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4	16	$6.251 \cdot 10^{-2}$	2.12	$3.027 \cdot 10^{-1}$	1.86
8	64	$2.241 \cdot 10^{-2}$	1.48	$2.146 \cdot 10^{-1}$	0.50
16	256	$7.914 \cdot 10^{-3}$	1.50	$1.518 \cdot 10^{-1}$	0.50
32	1024	$2.805 \cdot 10^{-3}$	1.50	$1.074 \cdot 10^{-1}$	0.50
64	4096	$9.935 \cdot 10^{-4}$	1.50	$7.613 \cdot 10^{-2}$	0.50

Space-time variational formulation

$$b(\widehat{u}_h + I^1_{h_t} Q^1_{h_x} \widetilde{u}_0, v_h) = \langle f, v_h \rangle_Q \quad \text{for all } v_h \in Y_h.$$

For the particular test function $v_h(x,t) = \psi_i^0(t)\varphi_\ell(x)$, this gives

$$\sum_{k=1}^{M_x} \int_{t_{i-1}}^{t_i} \frac{u_k^i - u_k^{i-1}}{h_t} \, dt \int_\Omega \varphi_k(x)\, \varphi_\ell(x)\, dx$$

$$+ \sum_{k=1}^{M_x} \int_{t_{i-1}}^{t_i} \left(u_k^{i-1} + \frac{t - t_{i-1}}{h_t}(u_k^i - u_k^{i-1}) \right) \, dt \int_\Omega \nabla_x \varphi_k(x) \cdot \nabla_x \varphi_\ell(x)\, dx$$

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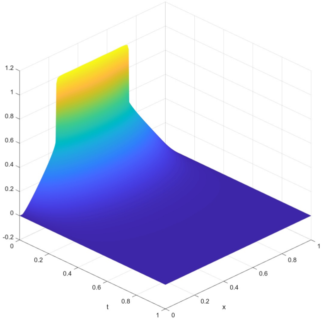
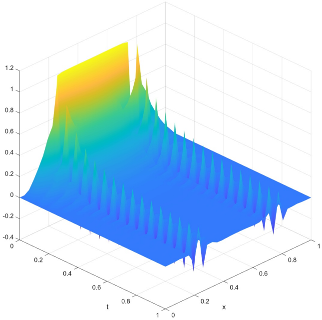
$$M_h(\underline{u}^i - \underline{u}^{i-1}) + \frac{1}{2} h_t K_h(\underline{u}^i + \underline{u}^{i-1}) = \underline{f}^i, \quad i = 1, \ldots, N_t.$$





Solution u

u_h for $N = N_t \cdot N_x = 961$

Strong oscillations in the numerical solution! $(h_t = h_x)$

O. Østerby: Five ways of reducing the Crank-Nicolson oscillations. BIT 43 (2003) 811–822.

Primal variational formulation

$$\int_0^T \int_\Omega \partial_t u \, v \, dx \, dt + \int_0^T \int_\Omega \nabla_x u \cdot \nabla_x v \, dx \, dt = \int_0^T \int_\Omega f \, v \, dx \, dt$$

Primal variational formulation

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Integration by parts in time

$$\begin{aligned}
\int_0^T \int_\Omega \partial_t u \, v \, dx \, dt &= \int_\Omega u \, v \, dx \Big|_0^T - \int_0^T \int_\Omega u \, \partial_t v \, dx \, dt \\
&= -\int_\Omega u_0 \, v(0) \, dx - \int_0^T \int_\Omega u \, \partial_t v \, dx \, dt \quad (v(T) = 0)
\end{aligned}$$

Primal variational formulation

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Adjoint variational formulation to find $u \in Y$ such that

$$b_T(u, v) := -\langle u, \partial_t v \rangle_Q + \langle \nabla_x u, \nabla_x v \rangle_{L^2(Q)} = \langle f, v \rangle_Q + \langle u_0, v(0) \rangle_{L^2(\Omega)}$$

is satisfied for all $v \in X_T := \{ v \in X : v(T) = 0 \}$.

Adjoint space-time FEM: Find $u_h \in Y_h := W^0_{h_t} \otimes V_{h_x}$ such that

$$b_T(u_h, v_h) = \langle f, v_h \rangle_Q + \langle u_0, v_h(0) \rangle_{L^2(\Omega)}$$

is satisfied for all $v_h \in X_{T,h} = \overline{W}^1_{h_t} \otimes V_{h_x}$, $\overline{W}^1_{h_t} = S^1_h(0, T) \cap H^1_{,0}(0, T)$.

Adjoint space-time FEM: Find $u_h \in Y_h := W^0_{h_t} \otimes V_{h_x}$ such that

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Discrete inf-sup stability condition

$$c \, \min\{1, h_t^{-1} \, h_x^2\} \, \|u_h\|_Y \leq \sup_{0 \neq v_h \in X_{,0,h}} \frac{b_T(u_h, v_h)}{\|v_h\|_X}$$

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Cea's lemma $(h_t = h_x^2)$

$$\|u - u_h\|_Y \;\leq\; c \inf_{v_h \in Y_h} \|u - v_h\|_Y \;\leq\; c \, \|u - Q^1_{h_x} Q^0_{h_t} u\|_Y \;\leq\; c \, h_x^s \, \|u_0\|_{\widetilde{H}^s(\Omega)}$$

Error estimate for $\|u - u_h\|_{L^2(Q)}$ follows similar as for the primal formulation.

Numerical example: smooth solution

$$\Omega = (-1,1), \quad T = 1, \quad h_t = h_x, \quad u(x,t) = \cos \pi t \sin \pi x$$

		$h_t = h_x$			
N_x	N_t	$\|u - u_h\|_{L^2(Q)}$	eoc	$\|u - u_h\|_Y$	eoc
2	2	$1.011 \cdot 10^{0}$		$3.106 \cdot 10^{0}$	
4	4	$4.508 \cdot 10^{-1}$	1.17	$1.967 \cdot 10^{0}$	0.66
8	8	$2.257 \cdot 10^{-1}$	1.00	$9.979 \cdot 10^{-1}$	0.98
16	16	$1.132 \cdot 10^{-1}$	1.00	$5.024 \cdot 10^{-1}$	0.99
32	32	$5.666 \cdot 10^{-2}$	1.00	$2.517 \cdot 10^{-1}$	1.00
64	64	$2.834 \cdot 10^{-2}$	1.00	$1.259 \cdot 10^{-1}$	1.00

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$$\Omega = (-1,1), \quad T = 1, \quad h_t = h_x, \quad u(x,t) = \cos \pi t \, \sin \pi x$$

N_x	N_t	$\|u-u_h\|_{L^2(Q)}$	eoc	$\|u-u_h\|_Y$	eoc
$h_t = h_x$					
2	2	$1.011 \cdot 10^{0}$		$3.106 \cdot 10^{0}$	
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8	8	$2.257 \cdot 10^{-1}$	1.00	$9.979 \cdot 10^{-1}$	0.98
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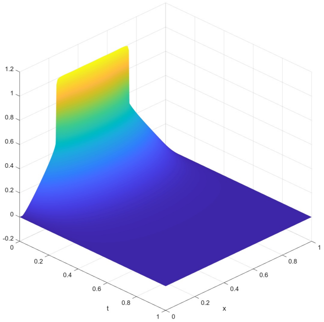
64	64	$2.834 \cdot 10^{-2}$	1.00	$1.259 \cdot 10^{-1}$	1.00
$h_t = h_x^2$					
2	4	$1.011 \cdot 10^{0}$		$3.106 \cdot 10^{0}$	
4	16	$2.196 \cdot 10^{-1}$	2.20	$1.420 \cdot 10^{0}$	1.13
8	64	$5.743 \cdot 10^{-2}$	1.94	$7.126 \cdot 10^{-1}$	0.99
16	256	$1.452 \cdot 10^{-2}$	1.98	$3.562 \cdot 10^{-1}$	1.00
32	1024	$3.641 \cdot 10^{-3}$	2.00	$1.781 \cdot 10^{-1}$	1.00
64	4096	$9.109 \cdot 10^{-4}$	2.00	$8.904 \cdot 10^{-2}$	1.00

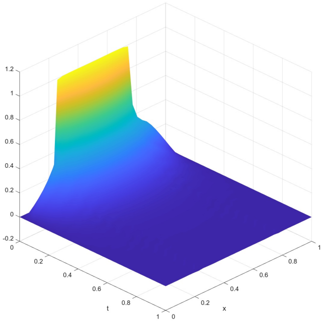
Numerical example: singular solution

$$\Omega = (-1,1), \quad T = 1, \quad h_t = h_x^2, \quad f(x,t) \equiv 0, \quad u_0(x) = \begin{cases} 1 & \text{for } |x| < \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

N_x	N_t	$\|u - u_h\|_{L^2}$	eoc	$\|u - u_h\|_Y$	eoc
2	4	$1.402 \cdot 10^{-1}$		$9.934 \cdot 10^{-1}$	
4	16	$6.058 \cdot 10^{-2}$	1.21	$3.224 \cdot 10^{-1}$	1.62
8	64	$2.023 \cdot 10^{-2}$	1.58	$2.211 \cdot 10^{-1}$	0.54
16	256	$7.123 \cdot 10^{-3}$	1.51	$1.569 \cdot 10^{-1}$	0.50
32	1024	$2.513 \cdot 10^{-3}$	1.50	$1.109 \cdot 10^{-1}$	0.50
64	4096	$8.877 \cdot 10^{-4}$	1.50	$7.861 \cdot 10^{-2}$	0.50



Solution u



u_h for $N = N_t \cdot N_x = 961$

No oscillations in the numerical solution! $(h_t = h_x)$

For $i = 1, \ldots, N_t - 1$:

$$
\begin{aligned}
-\sum_{k=1}^{M_x} \int_{t_{i-1}}^{t_{i+1}} u_k(t)\, \partial_t \psi_i^1(t)\, dt \int_\Omega \varphi_k(x) \varphi_\ell(x) dx \\
+ \sum_{k=1}^{M_x} \int_{t_{i-1}}^{t_{i+1}} u_k(t) \psi_i^1(t)\, dt \int_\Omega \nabla_x \varphi_k(x) \cdot \nabla_x \varphi_\ell(x)\, dx \\
= \int_{t_{i-1}}^{t_{i+1}} \int_\Omega f(x,t) \varphi_\ell(x)\, dx\, \psi_i^1(t)\, dt
\end{aligned}
$$

For $i = 1, \ldots, N_t - 1$:

$$-\sum_{k=1}^{M_x} \int_{t_{i-1}}^{t_{i+1}} u_k(t)\, \partial_t \psi_i^1(t)\, dt \int_\Omega \varphi_k(x)\varphi_\ell(x) dx + \sum_{k=1}^{M_x} \int_{t_{i-1}}^{t_{i+1}} u_k(t) \psi_i^1(t)\, dt \int_\Omega \nabla_x \varphi_k(x) \cdot \nabla_x \varphi_\ell(x)\, dx = \int_{t_{i-1}}^{t_{i+1}} \int_\Omega f(x,t)\varphi_\ell(x)\, dx\, \psi_i^1(t)\, dt$$

Crank-Nicolson scheme

$$M_h(\underline{u}^{i+1} - \underline{u}^i) + \frac{1}{2}\, h_t\, K_h(\underline{u}^{i+1} + \underline{u}^i) \,=\, \underline{f}^i$$

For $i = 0$:

$$- \sum_{k=1}^{M_x} \int_{t_0}^{t_1} u_k(t)\, \partial_t \psi_0^1(t)\, dt \int_\Omega \varphi_k(x) \varphi_\ell(x) dx$$

$$+ \sum_{k=1}^{M_x} \int_{t_0}^{t_1} u_k(t) \psi_0^1(t)\, dt \int_\Omega \nabla_x \varphi_k(x) \cdot \nabla_x \varphi_\ell(x)\, dx$$

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Implicit Euler scheme

$$M_h \underline{u}^1 + \frac{1}{2} h_t\, K_h \underline{u}^1 = \underline{f}^0 + \underline{u}^0$$

For $i = 0$:

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Implicit Euler scheme

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→ smoothing projection of the initial datum

R. Rannacher: Finite Element Solution of Diffusion Problems with Irregular Data. Numer. Math. 43 (1984) 309–327.

Conclusions

- Smooth data and regular solutions
  - primal formulation with $h_t \simeq h_x$
  - use of higher order polynomials in space and time
  - efficient iterative/direct solution of linear algebraic systems

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  - multiple smoothing steps
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- Application to other evolution problems (Stokes, Navier–Stokes, ...)
- Formulations in anisotropic Sobolev spaces $H^{1,1/2}(Q)$ using a (modified) Hilbert transform [OS, Zank: ETNA 2020]
- Coupled problems (elliptic-parabolic-hyperbolic)
- ...

24. Söllerhaus Workshop on

Fast Boundary Element Methods and Space-Time Discretization Methods

Kleinwalsertal, 24.–27.9.2026

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- 10. International Conference on Domain Decomposition Methods, Boulder, USA, 10.–14.8.1997.

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- 2/2006: Guest lecturer at VSB TU Ostrava

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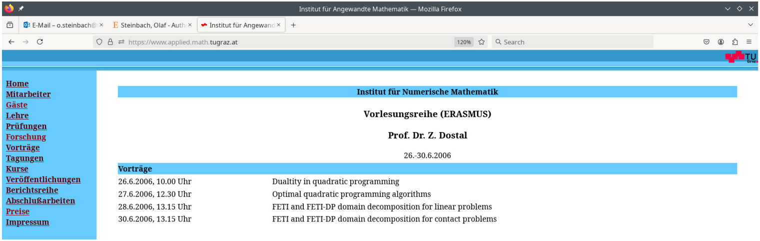
Institut für Numerische Mathematik

Vorlesungsreihe (ERASMUS)

Prof. Dr. Z. Dostal

26.-30.6.2006

Vorträge	
26.6.2006, 10.00 Uhr	Dualtity in quadratic programming
27.6.2006, 12.30 Uhr	Optimal quadratic programming algorithms
28.6.2006, 13.15 Uhr	FETI and FETI-DP domain decomposition for linear problems
30.6.2006, 13.15 Uhr	FETI and FETI-DP domain decomposition for contact problems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Happy Birthday, Zdenek!